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**Physics.** — *Circuits for the measurement of resistance.* (20th communication from the "VAN DER WAALS-Fund"). By P. GEELS and A. MICHELS. (Communicated by Prof. J. D. v. D. WAALS JR.)

(Communicated at the meeting of December 17, 1927).

From the calculations in the literature of circuits for resistance measurement, it is very difficult to obtain a good survey of the best circuit, of the maximum obtainable accuracy etc. This is a result of the use of, either a very special method of derivation for special cases, or a method which, although generally applicable, is so long that the physical meaning is obscured.

An attempt has therefore been made to find a method, which was comparative and applicable to all circuits in use and which, because of its simplicity, would enable the best circuit to be chosen without much calculation.

To this end use has been made of five rules, which have been applied singly by various writers, but which have never been used as a complete set. Long derivations resulting from the application of the general laws for the division of currents, may be avoided by the use of these rules.

These five rules are:

1. The current in a resistance  $g$  is equal to the algebraic sum of the currents found to exist in  $g$  when the electromotive forces in the resistance net are considered separately. (law of superposition).

This rule is a direct consequence of KIRCHHOFF's laws.

2. Every resistance  $R$  in a current net may be replaced by an *E.M.F.* equal to  $iR$  ( $i$  being the current in  $R$ ), without the currents in the net being altered.

This rule is also a direct consequence of KIRCHHOFF's laws.

SCHUSTER<sup>1)</sup> has pointed out that it was useful in the determination of the maximum sensitivity of a resistance measurement, but he made no special application of the rule.

3. The current  $i_g$  resulting from an *E.M.F.*  $E$  in  $R$  is equal to  $i_R$  resulting from the same *E.M.F.*  $E$  in  $g$ .

This rule has been suggested by MAXWELL<sup>2)</sup>, amongst others, and has been applied by F. E. SMITH to the calculation of the sensitivity of THOMSON's Bridge.<sup>3)</sup>

<sup>1)</sup> Phil. Mag. 1895. 15.

<sup>2)</sup> Electr. a. Magn. 1904. Vol. 1 page 401.

<sup>3)</sup> Brit. Ass. Report 1906. page 107.

4. If the conditions in a resistance net are such, that an *E.M.F.* in  $R$  leaves the link  $g$  without current, then the current  $i_g$ , resulting from a second arbitrary *E.M.F.* brought into the net, is independent of  $R$  (ROSEN's rule<sup>1)</sup>).

5. A new rule:

If the conditions in a resistance net are such, that the algebraic sum of two currents in  $g_1$  and  $g_2$  remains equal to 0 with an *E.M.F.*  $E$  in  $R$ , then the current  $i_{g_1} + i_{g_2}$  resulting from a second arbitrary *E.M.F.* brought into the net is independent of  $R$ .

This rule may be easily derived from the above.

As it is intended that the formulae derived should indicate which circuit offers the most sensitive possibility of measurement, it is obviously necessary to consider the galvanometers as used under the most suitable conditions. It will be supposed that the resistance of the windings for a given winding space may be chosen in the case of a needle galvanometer (e.g. *shielded galvanometer*), and that the magnetic field of a rotating coil galvanometer is so adjusted, as is the usual practice, that a deflection is obtained within the aperiodic limits.

In the first case the current sensitivity of the galvanometer is given by the formula,

$$\xi = \xi_0 \sqrt{g} \text{ } ^2)$$

where  $g$  represents the resistance of the galvanometer and  $\xi_0$  the sensitivity when  $g = 1$ .

In the second case

$$\xi = \xi_0 \sqrt{R_0} \text{ } ^2)$$

where  $R_0$  is the substitution resistance of the resistance net from the terminals of the galvanometer, plus the resistance of the galvanometer itself, and  $\xi_0$ , the galvanometer sensitivity when  $R_0 = 1$ .

The electrical sensitivity of a resistance measurement is determined by the following problem:

given a resistance  $R$ , how small may  $dR$  be made for the method of measurement to still distinguish between  $R$  and  $R + dR$ ?

Now, according to the second of the above rules, the effect of  $dR$  may be replaced by the effect of an *E.M.F.*  $e = idR$ , and theoretically it should be possible to make this any arbitrary amount for a given  $dR$  by increasing  $i$ . The value of  $i$  however, is limited by secondary conditions, chiefly the heating of the wire or Peltier effect.

This  $i$  must therefore occur as explicitly as possible in the formulae derived in order to facilitate their comparison. The other resistances of the circuit, usually consisting of resistance boxes, etc., will not as a rule be so restricted. Their temperature coefficient is small and they can be

<sup>1)</sup> Oefvers. k. Vetensk. Ak. Förhandl. 1887, page 203.

<sup>2)</sup> JAEGER. Messtechnik.

placed in oil baths (standard resistances) in order to diminish disturbing influences.

The derivation will also give, with the various methods of measurement, a means of obtaining a simple survey of the (already known) presence or absence of the influence of the leads and of the influence of eventual alterations in the resistances of these leads (e.g. temperature effects).

The above considerations may be applied to some of the more important circuits.

### WHEATSTONE's Bridge.

Commencing from the known conditions of equilibrium  $\frac{r_1}{r_2} = \frac{r_3}{r_4}$ , it is

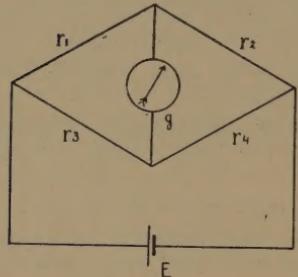


Fig. 1.

first necessary to calculate the current  $i_g$  resulting from an alteration  $dr_1$ .

According to rule 2,  $dr_1$  may be replaced by an E.M.F.,  $e = i_1 dr_1$ . From rules 1 and 4 applied successively,  $E$  and  $R$  may be left out of consideration. The E.M.F.  $e$  is then placed in  $g$  and  $i_1$  calculated (Rule 3). The required  $i_g$  is then obtained immediately

$$i_g = \frac{i_1 dr_1}{g + \frac{(r_1 + r_3)(r_2 + r_4)}{r_1 + r_2 + r_3 + r_4}} \times \frac{r_2 + r_4}{r_1 + r_2 + r_3 + r_4}$$

where  $\frac{(r_1 + r_3)(r_2 + r_4)}{r_1 + r_2 + r_3 + r_4} = \zeta$  the substitution resistance of the net from the point of view of the terminals of the galvanometer.

Substituting  $r_2 = nr_1$ ,  $r_3 = mr_1$  and  $r_4 = nm r_1$  in accordance with W. JAEGER, then

$$i_g = \frac{i_1 dr_1}{g + \frac{nr_1(m+1)}{n+1}} \cdot \frac{n}{n+1}.$$

For maximum sensitivity let  $i_1 = i_{\max.}$  ( $i_m$ ).

Let  $\varphi$  be the deflection of the needle galvanometer resulting from  $dr_1$ , then.

$$\varphi = \xi_0 \frac{i_m dr_1 \sqrt{g}}{g + \frac{nr_1(m+1)}{n+1}} \cdot \frac{n}{n+1} \quad \dots \quad (1)$$

From (1) it is seen that  $\varphi$  increases with increasing  $n$  and with decreasing  $m$ . Hence if  $n \gg 1$  and  $m \ll 1$  the best limiting value is given by

$$\varphi = \xi_0 \frac{i_m dr_1 \sqrt{g}}{g + r_1}.$$

also for a maximum it is found that  $g = \zeta$ , and in general

$$g = \frac{n r_1 (m + 1)}{n + 1}$$

whence

$$\varphi_{\max} = \xi_0 \frac{i_m dr_1}{2 \sqrt{r_1}}.$$

If  $\psi$  is the deflection of a moving coil galvanometer the equations become

$$\psi = \xi_0 \frac{i_m dr_1 \sqrt{g + \frac{n r_1 (m + 1)}{n + 1}}}{g + \frac{n r_1 (m + 1)}{n + 1}} \cdot \frac{n}{n + 1} = \xi_0 \frac{i_m dr_1}{\sqrt{g + \frac{n r_1 (m + 1)}{n + 1}}} \cdot \frac{n}{n + 1}. \quad (2)$$

and taking  $n \gg 1$  and  $m \ll 1$ , the limiting value becomes

$$\psi_{\max} = \xi_0 \frac{i_m dr_1}{\sqrt{g + r_1}}.$$

### THOMSON's Bridge.

THOMSON's Bridge is given as a first example of the extension of WHEATSTONE's Bridge in order to eliminate the resistance of the necessary leads.

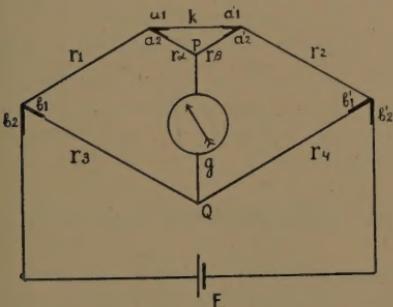


Fig. 2.

This is immediately seen on breaking

$g$  and investigating the potential difference between  $P$  and  $Q$ . If it is possible to make the ratios of the resistances with a sufficient order of accuracy then  $r_1$  may be expressed in terms of  $r_2$ ,  $r_3$  and  $r_4$ . The theory of the sensitivity of this circuit may also be simplified with the aid of the current laws.

$$\text{When } \frac{r_2}{r_3} = \frac{r_1}{r_2} = \frac{r_3}{r_4} \text{ then } i_g = 0.$$

Commencing from the condition that  $\frac{r_\alpha}{r_\beta} = \frac{r_1}{r_2} = \frac{r_3}{r_4}$ , the deflection of the galvanometer  $g$  when  $r_1 \rightarrow r_1 + dr_1$  is first calculated.  $dr_1$  may be replaced by an E.M.F.  $e = i_1 dr_1$ . As  $\frac{r_\alpha}{r_\beta} = \frac{r_1}{r_2} = \frac{r_3}{r_4}$ , an E.M.F.  $k$  and  $R$  will have no influence on  $i_g$ . It is therefore possible to choose arbitrary values for  $R$  and  $k$  for the calculation of  $i_g$  resulting from  $e$  in  $r_1$ ; therefore let  $R = \infty$  and  $k = 0$ .

THOMSON's Bridge then reduces to a very simple circuit from which it follows immediately that, as far as the sensitivity is concerned, the bridge is equivalent to a WHEATSTONE's Bridge where

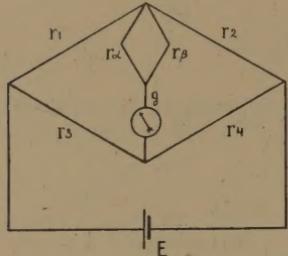


Fig. 3.

$g$  is increased by a resistance  $\frac{r_\alpha r_\beta}{r_\alpha + r_\beta}$  (see fig. 3).

From this it is also clearly seen what resistances must be chosen in order to obtain a maximum sensitivity of the bridge. As far as concerns the self induction,  $k$  may also be short circuited and  $R$  broken (Rule 4)

when a moving coil galvanometer is used. It is seen that  $r_\alpha$  and  $r_\beta$  have a prejudicial influence on the sensitivity of the measurements.

The formula may be simplified by JAEGER's method by substituting

$$r_\alpha = \mu r_1, \text{ whence } r_\beta = n\mu r_1 \text{ and } \frac{r_\alpha r_\beta}{r_\alpha + r_\beta} = r_1 \mu \left( \frac{n}{n+1} \right)$$

and equation (1) becomes

$$\varphi = \xi_0 \frac{i_m dr_1 \sqrt{g}}{g + r_1 (m+1) \frac{n}{n+1} + r_1 \mu \frac{n}{n+1}} \cdot \frac{n}{n+1}$$

$$= \xi_0 \frac{i_m dr_1 \sqrt{g}}{g + r_1 (m+\mu+1) \frac{n}{n+1}} \cdot \frac{n}{n+1}.$$

From a simple determination of the conditions for a maximum  $\varphi$  it follows that

$$g = \xi = r_1 (m + \mu + 1) \frac{n}{n+1}$$

whence

$$\varphi = \xi_0 \frac{i_m dr_1}{2 \sqrt{r_1 (m + \mu + 1) \frac{n}{n+1}}}$$

with the most favourable limit being when  $m \ll 1$ ,  $\mu \ll 1$  and  $n \gg 1$ .

which, of course, reduces to the same as that of the WHEATSTONE's Bridge, namely

$$\varphi_{max} = \xi_0 \frac{i_m dr_1}{2 \sqrt{r_1}}.$$

Equation (2) for a moving coil galvanometer becomes

$$\begin{aligned} \varphi &= \xi_0 \frac{i_m dr_1 \sqrt{g + r_1 (m + \mu + 1) \frac{n}{n+1}}}{g + r_1 (m + \mu + 1) \frac{n}{n+1}} \cdot \frac{n}{n+1} \\ &= \xi_0 \cdot \frac{i_m dr_1}{\sqrt{g + r_1 (m + \mu + 1) \frac{n}{n+1}}} \cdot \frac{n}{n+1} \end{aligned}$$

with the most favourable limit when  $m \ll 1$  and  $\mu \ll 1$  and  $n \gg 1$ ,

$$\varphi_{max} = \xi_0 \frac{i_m dr_1}{\sqrt{g + r_1}}.$$

When the secondary physical conditions (e.g. the influence of the leads) do not allow of the best circuit being obtained, then the formulae are a complete guide to the most satisfactory solution.

The following considerations refer to the exclusion of the leads. Alterations in  $a_1 a'_1 b_2 b'_2$  can have no influence on  $i_g$ , as an E.M.F. in  $k$  and  $R$  during the regulating of the resistances, has no effect on the galvanometer.

The influence of alterations in  $b_1$  and  $b'_1$  may be diminished by making  $r_3$  and  $r_4$  large, and alterations in  $a_2$  and  $a'_2$  may likewise be diminished by making  $r_\alpha$  and  $r_\beta$  large with respect to  $k$ , in other words by making  $i_\alpha$  and  $i_\beta$  small in respect to  $i_1$  (this is also at the expense of the sensitivity).

### *Second extension of WHEATSTONE's Bridge.*

A further extension of WHEATSTONE's Bridge, also serving to eliminate the influence of the leads, but less known than THOMSON's Bridge, (JAEGER only mentions this circuit) is given in Fig. 4.  $a_1 \dots b'_2$  again represent the leads of the measuring and comparison resistances  $r_1$  and  $r_3$  respectively.

When  $\frac{r_\alpha}{r_\beta} = \frac{r_1}{r_3} = \frac{r_2}{r_4}$  then, by the same reasoning as used with THOMSON's Bridge,  $i_g = 0$ . If it is possible to regulate the ratios of the resistances

with complete accuracy (this is not alone determined by  $i_g = 0$ ) then  $r_1$  may be expressed in terms of  $r_2, r_3$  and  $r_4$ .

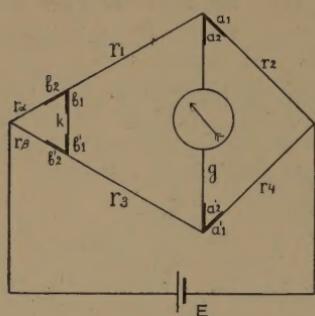


Fig. 4.

has no influence on  $i_g$ , and alterations in  $r_2$  and  $r_3$  may be diminished by making  $r_2 + r_3$  large compared to  $k$ .

The advantages of the elimination of the leads by the THOMSON Bridge and by the second extension of WHEATSTONE's Bridge may be combined and extended to a circuit such as in Fig. 5.

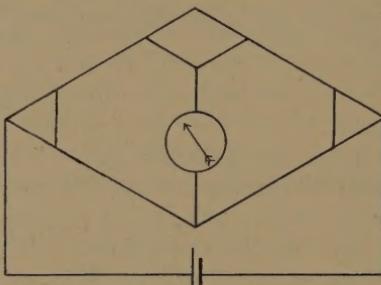


Fig. 5.

### Potentiometer.

The potentiometer in its simplest form is given in Fig. 6, and in practice  $r_6 = 0$  and in a nul-adjustment  $i_g$  also  $= 0$ . When  $r_1 \rightarrow r_1 + dr_1$  it is required to know the value of  $i_g$ .

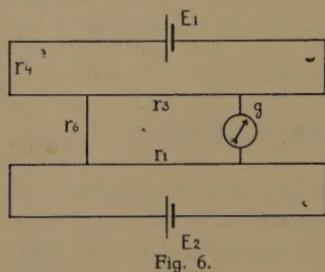


Fig. 6.

currents in shunts it is found at once that

$$i_g = \frac{i_1 dr_1}{g + \frac{r_3 r_4}{r_3 + r_4} + \frac{r_1 r_5}{r_1 + r_5}} \cdot \frac{r_5}{r_1 + r_5} \quad \dots \quad (3)$$

The sensitivity of this bridge may be obtained as before by substituting an *E.M.F.*  $e = i_1 dr_1$  for  $dr_1$ . Also from rule 4,  $R$  may be broken for the current in  $g$  caused by  $e$ . When  $k$  is made as small as possible (thus  $k = b_1 + b_2$ ) then it is at once seen that the sensitivity is practically the same as that of WHEATSTONE's bridge and therefore independent of  $r_\alpha$  and  $r_\beta$ . Reference is therefore made to the previous derivations. It is clear that alterations to  $b_1$  and  $b'_1$  will have no influence, as the *E.M.F.* in  $k$

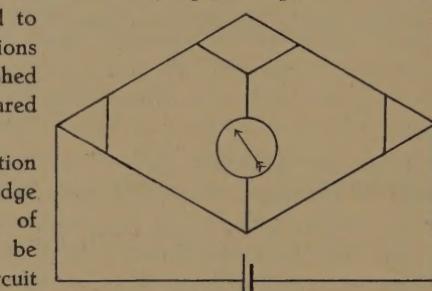


Fig. 5.

$dr_1$  may be replaced by an *E.M.F.*  $e = i_1 dr_1$ .  $E_1$  and  $E_2$  may be neglected in the calculation of  $i_g$ , which is then obtained by placing  $e$  in  $g$  and calculate the current in  $r_1$ . From the known division of

in which  $\frac{r_3 r_4}{r_3 + r_4} + \frac{r_1 r_5}{r_1 + r_5}$ , according to the definition, may be represented by  $\zeta$ , the substitution resistance of the net outside  $g$ . In order to obtain a greater sensitivity of the measurement and for other practical reasons  $r_5 \gg r_1$ . As a first approximation, equation 3 then becomes

$$i_g = \frac{i_1 dr_1}{g + r_1 + \frac{r_3 r_4}{r_3 + r_4}}$$

If a needle galvanometer is used

$$\varphi = \xi_0 \frac{i_1 dr_1 \sqrt{g}}{g + r_1 + \frac{r_3 r_4}{r_3 + r_4}}.$$

The necessary condition of a maximum for  $\varphi$  is found to be that

$$g = r_1 + \frac{r_3 r_4}{r_3 + r_4}$$

whence

$$\varphi = \xi_0 \frac{i_1 dr_1}{2 \sqrt{r_1 + \frac{r_3 r_4}{r_3 + r_4}}}.$$

When a moving coil galvanometer is used

$$\psi = \xi_0 \frac{dr_1 \sqrt{g + r_1 + \frac{r_3 r_4}{r_3 + r_4}}}{g + r_1 + \frac{r_3 r_4}{r_3 + r_4}} = \xi_0 \frac{i dr_1}{\sqrt{g + r_1 + \frac{r_3 r_4}{r_3 + r_4}}}.$$

For the sensitivity of the measurement it is therefore necessary to make  $\frac{r_3 r_4}{r_3 + r_4}$  as small as possible. But the potential difference in  $r_3$  is equal to that in  $r_1$ , and hence if  $i_3$  is not to be made too large (in relation to current alterations and temperature influences) then  $r_3$  has a minimum value. Also  $r_4$  cannot be made too small, as it is used to adjust the potential difference. Hence it is seen this in general  $\frac{r_3 r_4}{r_3 + r_4}$  will not differ much in order from  $r_1$ .

The known properties of the potentiometer as far as concerns the exclusion of the influence of alterations to the potential leads again follow from the general principles developed.

KOHLRAUSCH's *Differential method with over-lapping shunts*.

In the calculation of the sensitivity of this circuit it is understood

that a differential moving coil galvanometer is used. For the differential needle galvanometer reference is made to W. JAEGER<sup>1</sup>).

It will be understood that, with a current sensitivity  $\xi$  of the differential galvanometer, a deflection  $\varphi$  will be obtained, when the algebraic sum of the galvanometer currents  $i_{g_1}$  and  $i_{g_2}$  is equal to unit current strength, on the supposition that equal currents in the two coils produce equal torques ( $i_{g_1}$  and  $i_{g_2}$  are given the same sign as they both give couples in the same direction).

If the galvanometer is outwardly closed by a resistance, then the rotation of coil will produce induction currents in the windings, which will result in a damping of the deflection. For accurate measurements, care must be taken that the magnetic field is so adjusted that a deflection is obtained within the aperiodic limit. If the two galvanometer windings are independently closed by shunt  $r$ , then, in complete agreement with a normal galvanometer,  $\xi$  will be proportional to  $\sqrt{r+g}$ .

Let  $r+g = R_0$  and  $\xi = \xi_0$  when  $h_0 = 1$  then  $\xi = \xi_0 \sqrt{R_0}$ .

Figure 7 gives KOHLRAUSCH's circuit, the special current commutation being omitted as this is of no importance for the calculation of the sensitivity (galvanometer currents of the same sign are represented in the figure).

When the measuring resistances  $r_1$  and  $r_2$  are equal, then, with an E.M.F. in  $r_2$  and  $r_4$  the deflection of the differential manometer  $= 0$  (JAEGER showed that, when  $g_1 \neq g_2$  similar currents in  $g_1$  and  $g_2$  resulted in an unbalanced couple, but the deflection  $\psi$  remained constant after commutating).

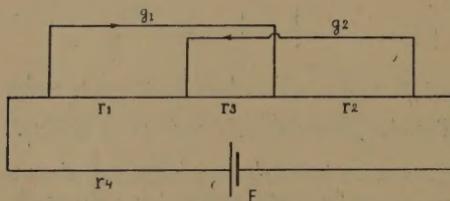


Fig. 7.

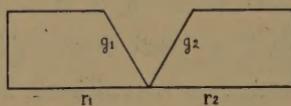


Fig. 8.

For the sensitivity of the measurement it is of importance to know  $\psi$  when  $r_1 \rightarrow r_1 + dr_1$  ( $\psi$  being originally zero). The alteration in resistance  $dr_1$  may be replaced by an E.M.F.  $e = i_1 dr_1$ . From rule 2,  $E$  may be omitted in the calculation of  $i_{g_1} + i_{g_2}$ . Then from rule 5,  $r_4$  may be broken and  $r_2$  short circuited, the galvanometer remaining within the aperiodic limits. The induction E.M.F.'s produced by the rotation of the coil have, according to rule 5, equal influences in  $i_{g_1} + i_{g_2}$ , and

<sup>1</sup>) Messtechnik.

thus cause the original damping. The circuit may therefore be simplified to Figure 8. It is then found immediately that

$$i_{g_1} + i_{g_2} = i_{g_1} = \frac{e}{r_1 + g} = \frac{i_1 dr_1}{r_1 + g}$$

whence

$$\psi = \xi_0 \frac{i_1 dr_1}{r_1 + g} \sqrt{r_1 + g} = \xi_0 \frac{i_1 dr_1}{\sqrt{r_1 + g}}.$$

**Physics. — *The Behaviour of thick walled Cylinders under High Pressures.***

By A. MICHELS. (23<sup>rd</sup> Communication of the "VAN DER WAALS Fund".) (Communicated by Prof. J. D. VAN DER WAALS Jr.)

(Communicated at the meeting of May 26, 1928).

**§ 1. *Introduction. The importance of research at high pressures.***

After metals have been subjected to a mechanical treatment by the application of either a compression, tension or shear, in which the yield point has been passed, they show, in general, an alteration in their physical properties and their strength is found to have increased.

There is reason to believe that the effect of the mechanical treatment is a complicated phenomenon, and it is suggested that the whole phenomenon be called "Strengthening" <sup>1</sup>).

The influence of strengthening has not as yet been introduced into the usual theories of thick walled tubes. The question therefore arises as to whether it is possible to increase the range of obtainable pressures by a proper application of this phenomenon.

BRIDGMAN has made measurements up to pressures of 20.000 atms. and just above these pressures, the study of certain physical properties becomes of great importance.

The theory of BORN, for example, has indicated that the forces in the space lattice of certain ionic crystals such as NaCl, are of the order of 200 Kilo per mm<sup>2</sup>, which corresponds to a pressure of 20.000 Atms. <sup>2</sup>) and it would therefore be of great interest to investigate the influence of forces, which are of the same order and greater than those in the space lattices.

The study of other physical properties, such as the compressibilities of liquids and solids, under pressures higher than so far obtained, is also of great interest. The researches of P. W. BRIDGMAN have shown that the compressibility of liquids may be divided in two parts: a liquid is usually very compressible for the first 2000—3000 Atms. but above this pressure the compressibility falls off rapidly and becomes small and constant. BRIDGMAN suggests that the molecules become close packed during the first part of the compression, and that, when this packing is completed, only the molecules themselves can be compressed. In agreement with this hypothesis, solids do not show the first part of the compressibility.

It is now of importance to see whether the molecules are really compressible, and if so what is the final result. Will there be an actual

<sup>1</sup>) Ann. d. Physik, (4) **85**, 770 (1928).

<sup>2</sup>) See also the publications of A. JOFFÉ c.s.: Zs. f. Phys. **31**, 286, 1924; Zs. f. Phys. **35**, 442, 1926; Proc. of the Int. Congress of Applied Mechanics (1925) 64.

deformation of the electron paths? In this connection contact is thus made with geo- and astro-physical problems<sup>1).</sup>

Another important research is the continuation of the curve found by BRIDGMAN for the influence of pressure on the electrical conductivity of certain metals. The resistance of Potassium, for example, is reduced to a quarter of its original value at a pressure of 12,000 Atms. and the resistance tends to continue to decrease. Will a minimum be found as in the case of Caesium or will a supra-conductive state be reached? It is also possible that the curve will be found to be asymptotic to the pressure axis, but no sign of a minimum can be seen in the curves already obtained.

Something similar is to be found in the thermal conductivity of liquids at 12,000 Atms. and BRIDGMAN has found that the thermal conductivity of certain liquids at this pressure is three times that at 1 atmosphere.

A summary of the phenomena in which study at high pressures is of great interest, was published three years ago by P. W. BRIDGMAN<sup>2).</sup>

*§ 2. The stresses in the walls of thick cylinders under high pressures.*

The stress curve in the material of a thick walled cylinder, when placed under pressure is often depicted as shown in Fig. 1.

When the maximum stress, which in this case lies inside the cylinder, surpasses the yield point, yielding will occur with a corresponding deformation and the stress curve will be given by Fig. 2. The material is

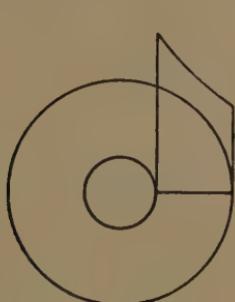


Fig. 1.



Fig. 2.

strained above the yield point between A and B, but remains below it between B and C. When the breaking stress is passed at A, the material will break.

It will not be possible to apply this explanation if the strengthening is taken into account.

It is to be expected that the strengthening will be determined by

<sup>1)</sup> See P. W. BRIDGMAN: "The breakdown of atoms at High Pressures". Phys. Rev. (2) 29, 180 (1927).

<sup>2)</sup> Certain Aspects of High Pressure Research. Journ. Franklin. Inst. 200, 147 (1925).

conditions under which the yielding it occurs. A homogeneous strengthening is to be expected only when the yielding of each point of the wall occurs under the same stresses.

Our knowledge of the mechanics of breaking is not sufficient to determine the stresses under which the material begins to yield and then to break, but some indication may be obtained of what is happening in the walls of a tube strained above the yield point.

Assume that the material has yielded to the point  $B$  (Fig. 2).

The radial pressure at  $B$  can be considered as a result of the forces acting from the middle, or it may be calculated from the forces of reaction acting from the outside of the point considered.

As, however, yielding has occurred in the inner layers, the forces can no longer be calculated from the first conception, and resource must be made to the second.

The division of the forces in the annular ring outside  $B$  may be calculated by the same method as for a tube, which has never been strained above the yield point. For this case the tangential and radial forces are given by the expressions : <sup>1)</sup>

$$S_r = C \left( \frac{1}{\varrho^2} + \frac{1}{R^2} \right) \quad S_t = -C \left( \frac{1}{\varrho^2} - \frac{1}{R^2} \right)^{1/2} \dots \dots \quad (I)$$

where  $C$  is an integration constant, which in the case considered is given by

$$C = p \frac{R^2 r^2}{R^2 - r^2} \quad \begin{aligned} \varrho &= \text{radius of observed point.} \\ p &= \text{pressure.} \\ 2R &= \text{external diameter.} \\ 2r &= \text{internal diameter.} \end{aligned}$$

These expressions will hold for the forces at a given point until the yield point is reached.

Although it is not known which of the forces (or what function of the forces) must reach a critical value before yielding occurs, it is possible to consider the problem further by making one of two simple assumptions.

If it is assumed that yielding is determined by the value of  $S_t$ , then it is simple to express the radial force at the moment when yielding begins, as a function of  $\varrho$  by eliminating  $C$ :

$$S_r = - \frac{R^2 - \varrho^2}{R^2 + \varrho^2} S_{tk}$$

where  $S_{tk}$  is the critical tangential tension which determines the yield point. As a first approximating  $S_{tk}$  may be considered constant. Fig. 3 then gives the radial force  $S_r$  at each point of the cross section at the moment that the tangential force reaches the value  $S_{tk}$  at that point, and yielding commences. The strengthening with therefore not occur from a uniform

<sup>1)</sup> See A. MICHELS. Ann. d. Physik, (4) 73, 617, (1924). These values follow simply from the expression  $S_t - S_a - x \frac{dS_r}{dx} = 0$ , where  $S_a = 0$ .

$S_r$  and will therefore not be expected to be uniform over the whole diameter.

Preliminary experiments have justified these suppositions. The strengthening probably decreases in extent towards the outside of the cylinder.

If the idea of a "critical breaking force" <sup>1)</sup> is introduced for every point of the tube, as that force which cannot be surpassed even in the strengthened material without a break occurring, then this critical force will be dependent on the degree of strengthening.

The degree of strengthening and the value of the internal forces cannot be calculated from material constant data, and therefore it is impossible to describe the exact behaviour of the phenomena.

Figs. 1 and 2 may be modified to represent the forces in the material at a determined moment just before a break occurs. As the strengthening at A will be greater than at B, the difference in slope between A and B will be still greater. If the strengthening is continued until C the curve will become continuous.

If it is assumed that this has actually happened, then the curve of the critical breaking force after the strengthening may be represented by C—D in Fig. 4, and that of the forces present by A—D. In this case the

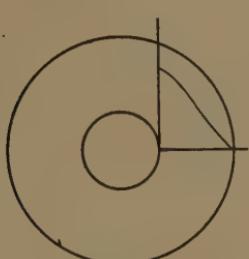


Fig. 3.

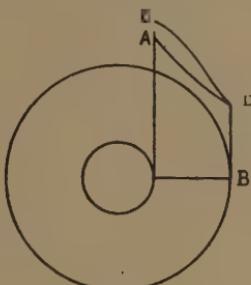


Fig. 4.

critical force will be surpassed on the outside before it is on the inside and a cylinder, in which this division of the stresses is present, will burst from the outside.

A division of the stresses as supposed above will be expected in a material with a large difference between the breaking and yielding points. The yielding can then continue for a longer time in the inside and can reach the outer surface.

On the other hand a material with a division of forces such as shown in Fig. 5 will burst from the inside. This is therefore to be expected in a material with only a small difference between the breaking and yielding points and a small elongation.

<sup>1)</sup> Here also it is not possible to decide whether this is due to a tensile or to a shear force, owing to the lack of knowledge of the theory of breaking.

It is also possible for both phenomena to occur almost, or exactly, at the same time as shown in Fig. 6.

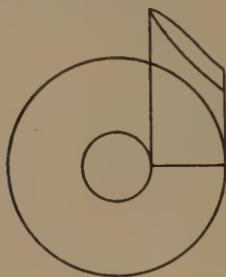


Fig. 5.

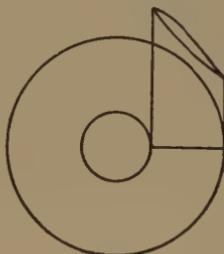


Fig. 6.

The second assumption that could have been made, is that the shear-force is the determining factor. A similar derivation may be made or the assumption that yielding occurs when the shear force exceeds a certain critical value  $S_{sk}$ .

Since the maximum shear-force is half the difference between the two main stresses, it may be represented as :

$$S_{s.m.} = \frac{1}{2} (S_r - S_t)$$

The derivation may be then made in an exactly similar manner to that given above.

The elimination of  $C$  and the substitution in (1) gives for the axial stress under which shearing occurs :

$$S_r = S_{sk} \left( 1 - \frac{\rho^2}{R^2} \right) \dots \dots \dots \quad (II)$$

which gives a curve as shown in Fig. 7.

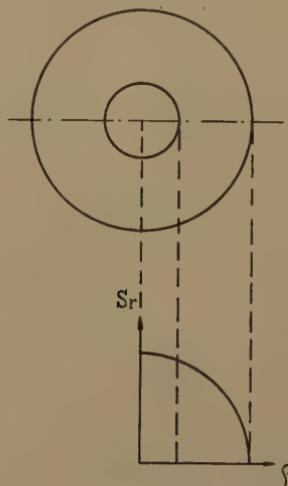


Fig. 7.

The most satisfactory material for the obtaining of high pressures must therefore be such that the strengthening is as large and can work as far as possible through the material. A material of sufficient critical strength in which the forces are distributed as in Fig. 6 will therefore be the best for high pressure construction.

### § 3. *Experimental.*

Certain experiments have been made on the basis of the above considerations<sup>1</sup>).

As so little was known of the degree of strengthening, the exact dimensions and properties of the materials used will not be given as the experiments were developed as they were made.

A commencement was made with a cylinder such as Fig. 8 of a material which had a large difference between the yielding and breaking points and a big elongation and which was therefore expected to burst from the outside.

The cylinder was closed at one end and the whole carefully polished.



Fig. 8.

A piston was ground in with as much play as would ensure that the piston would not jam in the cylinder, owing to the radial expansion when pressure was applied. The cylinder was filled with as thin an oil as possible and the piston pushed into the cylinder by means of a calibrated hydraulic press. In order to reduce the leak, the first 6.000 Atms. were applied as quickly as possible. The pressure increased the viscosity of the oil and therefore diminished the leak. The pressure was then increased until the material began to yield. In order to assure that the friction was not too great, the load was often decreased in

order to see if the piston followed the movement of the press. In this way it was shown that the friction did not rise above 2 %. As the load was known to 1 % the pressure could be calculated accurately enough from the load and the diameter of the piston.

A break occurred on the outside of the cylinder at 29.000 Atms., the burst being 2 mm wide (Fig. 10).

Owing to the expansion of the material, the pressure fell and the cylinder was not burst through as was shown by the fact that it still held a pressure of 13.000 Kilo. A material was then chosen with the same yielding and breaking limits, but with a somewhat smaller elongation and a break was obtained as shown in Fig. 11.

<sup>1</sup>) Shortly after the above had been written, the author's attention was drawn to a publication of P. W. BRIDGMAN, in the Proceedings of the 2nd International Congress for Applied Mechanics Zurich, 1926, in which he states that he has reached 40.000 Kilo per  $\text{cm}^2$  in steel tubes. He also mentions that he obtained a burst on the outside with some specimens of steel, but he gives no explanation of this.

A material of greater elongation gave a burst as shown in Fig. 12. Fig. 9 gives a case in which the break occurred at both sides at once. Such a material must give the highest pressures.



Fig. 9.

Fig. 10.

By raising the initial strength with the same elongation a material was obtained which held a pressure of 35.000 atms.

Another difficulty then developed, the solution to which has only just come into sight.



Fig. 11.

Fig. 12.

Up to the present the material of the piston had been chosen arbitrarily and a hardened chrome-nickel steel used, but it appeared that the high pressure load was becoming too great for this steel and several times the piston broke at 35.000 Atms. into small pieces 2—3 mm<sup>3</sup> in size. This was not the result of a bending force, for, although the piston was only 1½ cm out of the cylinder, it was broken over the whole length and therefore over the six and more cms in the cylinder.

It is not possible to predict how high a pressure will be obtainable.

It appears from our results that to obtain the highest pressures the greater the ratio  $\frac{R}{r}$ , the greater must be the difference between the yielding and breaking points. The few experiments made do not yet justify the communication of a functional relation, and it is therefore not yet possible

to say which steels are the best for high pressure construction. As has been mentioned above, it has not been found possible to burst the later cylinders but it is hoped that it will be possible to communicate shortly some more definite facts.

From the experiments it appears, as may be seen from the photographs, that the bursts from the inside are in the nature of shears, whilst those from the outside are due to tension. This may be dependent on the fact that in the inside where the bore is small compared to the outside diameter,  $S_{smax} = S_t$  whilst on the outside  $S_{max} = 1/2 S_t$ .

This result may perhaps bring further light on to the theory of the bursting.

The author is indebted to the Minister of war and to the Director of the Artillery Arsenal at Hembrug for permission to carry out these experiments in the Metallographic Laboratory of the Arsenal.

He is also indebted to D. VAN DEN BERG and to Dr. P. SCHOENMAKER of the Laboratory for their advice and collaboration.

The experimental cylinders were prepared in this Laboratory.

**Chemistry. — Osmosis of ternary liquids. Experimental part III.** By F. A. H. SCHREINEMAKERS and B. C. VAN BALEN WALTER.

(Communicated at the meeting of May 26, 1928).

In the preceding communications (Exp. I and II) we have discussed the apparent osmosis of some systems; now we shall consider their real osmosis, viz. the directions in which the substances pass through the membrane, etc.

First we shall briefly indicate in what way we have determined the point of intersection  $s_0$  of two conjugated chords. For this purpose we imagine the points  $a$  and  $b$  on the one branch of a path and the points  $a'$  and  $b'$  on the other branch. We represent the composition of the liquid  $a$  by:

$x_a$  quant. of  $X + y_a$  quant. of  $Y + (1 - x_a - y_a)$  quant. of  $W$   
and that of the liquid  $a'$  by:

$x'_a$  quant. of  $X + y'_a$  quant. of  $Y + (1 - x'_a - y'_a)$  quant. of  $W$

In order to represent the composition of the liquids  $b$  and  $b'$  we imagine the index  $a$  to be replaced by  $b$ .

The equation of the straight line, running through  $a$  and  $b$  is:

$$\frac{y - y_a}{y_a - y_b} = \frac{x - x_a}{x_a - x_b} \quad \dots \dots \dots \dots \dots \quad (1)$$

We now put:

$$\mu = -\frac{y_a - y_b}{x_a - x_b} \quad h = \mu x_a + y_a \quad \dots \dots \dots \quad (2)$$

We now may substitute (1) by:

$$y = -\mu x + h \quad \dots \dots \dots \quad (3)$$

The equation of the line  $a' b'$  is:

$$\frac{y - y'_a}{y'_a - y'_b} = \frac{x - x'_a}{x'_a - x'_b} \quad \dots \dots \dots \quad (4)$$

We now put:

$$\mu' = -\frac{y'_a - y'_b}{x'_a - x'_b} \quad h' = \mu' x'_a + y'_a \quad \dots \dots \dots \quad (5)$$

We now may substitute (4) by:

$$y = -\mu' x + h' \quad \dots \dots \dots \quad (6)$$

The equations (3) and (6) now define the  $x$  and  $y$  of the point of intersection  $s$  of the two chords; we find:

$$x = \frac{h - h'}{\mu - \mu'} \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (7)$$

after which we find  $y$  from (3) or (6).

If we represent the angle formed by the chord  $a b$  and the negative direction of the  $X$ -axis by  $\varphi$  then  $\mu = \operatorname{tg} \varphi$ ; besides we see that  $h$  is the part cut off the  $Y$ -axis by the chord  $a b$ . For  $\mu'$  and  $h'$  the same is valid with respect to the chord  $a' b'$ .

We also see that we may substitute the index  $a$  by  $b$  in the values of  $h$  and  $h'$  of (2) and (5).

Now we began by determining  $\mu$  and  $\mu'$  and afterwards  $h$  and  $h'$  with the help of (2) and (5), afterwards  $x$  with the help of (7) and  $y$  with (3) or (6); then we knew the composition:

$x$  quant. of  $X + y$  quant. of  $Y + (1-x-y)$  quant. of  $W$   
of the diffused mixture and consequently the position of the point  $s_0$ .

We shall just illustrate this deduction with an example; for this purpose we take the points 5 and 6 of the path of system V; we find it drawn schematically in fig. 1 Exp. I. It follows from table V:

$$\begin{array}{llll} x_5 = 7.312 & y_5 = 7.620 & x'_5 = 19.347 & y'_5 = 1.070 \\ x_6 = 9.188 & y_6 = 7.255 & x'_6 = 17.627 & y'_6 = 1.408 \end{array}$$

in which all concentrations are expressed in procents. From (2) and (5) follows:

$$\mu = 0.19456 \quad \mu' = 0.19651$$

and afterwards for  $h$  and  $h'$ :

$$h = 9.043 \quad h' = 4.872.$$

With the aid of (7) we now find:

$$x = -2139.$$

From (3) or (6) now follows:  $y = 425$ . As the concentrations have been expressed in procents, the  $W$ -amount is not  $1-x-y$ , now, but  $100-x-y$  or  $100+2139-425=1814$ . Consequently the composition of the diffused mixture is:

$$-2139\% X + 425\% Y + 1814\% W \quad \dots \quad \dots \quad \dots \quad (8)$$

The diffused mixture is situated, therefore, in field IV (fig. 1, Gen. III).

From the position of this point  $s_0$  with respect to the points 5 and 6, it appears that liquid 5 gives off this mixture; so we know that on part 5.6 of the path the mixture diffuses towards the right.

As the mixture contains negative quantity of  $X$  and a positive quantity of  $Y$  and  $W$ , this results for the directions, in which the substances pass through the membrane.

$$\xleftarrow{X} \quad \xrightarrow{Y} \quad \xrightarrow{W} \quad \dots \quad \dots \quad \dots \quad \dots \quad (9)$$

Now we can also find the composition of the diffused liquid. For it appears from (8) that the substances  $X$ ,  $Y$  and  $W$  have diffused through the membrane in the ratio:

$$2139 : 425 : 1814$$

If we now consider only the quantities and not the directions we see that:

$$2139 + 425 + 1814 = 4378$$

quantities have passed through the membrane; consequently the composition of the diffused liquid is:

$$48.9\% X + 9.7\% Y + 41.4\% W \dots \dots \dots \quad (10)$$

Consequently we find: the liquid which has passed through the membrane in the time during which the system passes along part 5.6 of the path, consists of 48.9% of  $X$  which diffuse towards the left, and of 9.7% of  $Y$  and 41.4% of  $W$  which diffuse towards the right. So a greater volume goes through the membrane towards the right than towards the left.

It appears from the values of  $\mu$  and  $\mu'$  in (2) and (5) that numerator and denominator are usually small; a small mistake in the determination of the concentrations may sometimes cause a large mistake in  $\mu$  and  $\mu'$  and consequently in the position of the point  $s_0$ .

For this reason the compositions of the left and the right liquids have been determined as accurately as possible. It appeared from many duplo-determinations that the penultimate of the decimals given is absolutely correct. Yet in some cases the influence may be great, viz. when two points are situated close to one another and especially when the two chords approach each other closely towards the end of the osmosis.

The results of the calculations may be found in the tables and schemes of the preceding communications Exp. I and II.

In the schemes we find the directions in which the substances have passed through the membrane on the different parts of the path.

In the tables we find for each part of the path:

a. the field in which point  $s_0$  is situated, and the direction in which this mixture has passed through the membrane.

b. sub  $X$ ,  $Y$  and  $W$  the composition of the diffused liquid.

We find them for part 1.2 of a path with determination n°. 2; for part 2.3 with determination n°. 3; etc.

The compositions of the diffused mixtures have not been indicated in the tables, as they have no meaning by themselves; of course the liquids situated in field I have the same composition as the diffused liquid.

In a preceding communication (Gen. V) we have seen that we have to know the points  $s_l$  and  $s_r$  viz. the compositions of the mixtures  $L_l$  and

$L_r$ , in order to learn to know the real osmosis accurately. In order to find them, we must, however, know not only the compositions of the liquids on the right and the left side of the membrane, but also their quantities. Our experiments, however, have until now not enabled us to define this quantity with sufficient accuracy; in our considerations we have to limit ourselves to point  $s_0$ .

This point  $s_0$ , however, as we have previously (l.c.) seen, only represents an approximate value of the mixtures  $L_l$  and  $L_r$  diffusing in reality; for this reason the composition of the diffused liquid has been indicated in the tables in one decimal only.

Yet the  $s_0$ -scheme deduced from point  $s_0$  generally indicates the directions in which the substances and the mixture pass through the membrane; we are shortly going to refer to the transition-cases, which have already been (l. c.) discussed before.

Towards the end of the osmosis the  $s_0$ -scheme, even when still in existence at the time is no longer dependable (l.c.); to this we may add as we have seen above, that the mistake in determining point  $s_0$  may be big then. In some schemes also the horizontal arrows with the last parts of the path have been omitted for this very reason.

Notwithstanding all those difficulties of which we ourselves are only too well aware, we yet think ourselves justified in briefly discussing the real osmosis of these systems. This discussion, however, should be looked upon only as initial orientation for further investigations in this field.

If in the schemes we pay attention to the signs  $>$  and  $<$ , which indicate whether the concentration of a substance on the left side of the membrane is larger or smaller than on the right side, and to the horizontal arrows which indicate the directions in which the substances pass through the membrane, we see that as well positive as negative osmosis occurs; in order to facilitate the survey, an asterisk has been placed with the horizontal arrows, which indicate a negative osmosis. We see from these schemes among other things the following:

the substance  $X$  diffuses in all systems in positive direction; II—V and in VII, however, also in negative direction;

the substance  $Y$  diffuses in all systems in positive direction; in III and IX, however, also in negative direction;

the substance  $W$  diffuses in all systems as well in positive as in negative direction.

As, however, the  $s_0$ -scheme is no more to be depended upon towards the end of the osmosis, we shall cancel all determinations in which the point  $s_0$  is situated within field I; these parts of the path have been indicated in the schemes by the sign 0. Then we find:

the substance  $W$  diffuses in all systems in positive direction, but in I—III and VI—XII also in negative direction. We then find examples, as well of normal-normal-negative as of anormal-anormal-negative  $W$ -osmosis.

Consequently we may conclude from this that in reality besides positive-also negative osmosis occurs; this is what might be expected from a theoretical point of view also.

We shall consider some of the systems a little more in detail. First we take system XII the path of which has been represented schematically in fig. 1 Exp. II. It appears from the table that the mixture  $s_0$  is situated in field IV during the total osmosis and goes towards the right. Consequently during all the osmosis the substance  $X$  goes towards the left (comp. scheme XII) and the substances  $Y$  and  $W$  towards the right.

Although, therefore, the water diffuses towards the left side during the whole of the osmosis, we yet have a positive osmosis on the part 1.6 of the path and on part 6.9 a negative  $W$ -osmosis; the reason for this is that the  $W$ -amount on part 1.6 is larger on the left side than on the right side, but on part 6.9 it is smaller on the left side than on the right side.

It appears from table V that the mixture  $s_0$  of the part 1.6 of the path of system V (comp. fig. 1 Exp. I) is situated in field IV and diffuses towards the right; for part 6.8 of path  $s_0$  is situated in field VII and diffuses towards the left. Consequently for the whole of part 1.8 obtains, as has been indicated also in the scheme, that  $X$  diffuses towards the left and  $Y$  and  $W$  towards the right (comp. Gen. V for the directions in which the mixtures  $L_l$  and  $L_r$  can diffuse at the moment that  $s_0$  passes from field IV towards the opposite field VII).

The mixture  $s_0$  of part 7.8 of the path is situated in field VII, that of part 8.9, however, in field VI; so in the vicinity of point 8 (viz. somewhere between 7 and 9) the mixture comes somewhere on the prolongation of the side  $XY$ . As the mixture now does not contain  $W$ , no  $W$  passes through the membrane at this moment. We may represent the directions in which the water moves, by:

$$\xrightarrow{W} \xrightarrow{W} \xleftarrow{W} \dots \dots \dots \quad (11)$$

The first arrow obtains when  $s_0$  is still in field VII; the last, when  $s_0$  has arrived in VI; the dash obtains for the moment that  $s_0$  comes on the side  $XY$  and, therefore, no  $W$  diffuses.

Previously (Gen. V), however, we have already seen that in- and in the vicinity of such a point of transition small quantities of  $W$  yet will go towards the left or the right side, or are taken in or given off by the membrane. The  $s_0$  scheme does not indicate these transitions; as we have seen above, (l.c.) we have to replace it in this case by a scheme, consisting of several groups.

But even, when the  $s_0$ -scheme should indicate the correct directions

for the  $W$ -amount, the transition is not so simple as it has been represented above. For on the part 1.8 of the path the  $W$ -amount on the left side is larger than on the right side; on part 8.9, however, it is smaller on the left side than on the right side (comp. scheme V); in the vicinity of point 8, therefore, is situated a point, where both liquids have the same  $W$ -amount. Then, if we do not consider the direction in which the water diffuses, we have the symbol:

$$*\downarrow = \uparrow* \quad \dots \quad (12)$$

Above, however, we saw that in the vicinity of point 8 also a transition, such as has been indicated in (11), is situated. If in (12) we now also indicate the direction, in which the water passes through the membrane, we see that one of the three symbols:

$$\xrightarrow{*} \downarrow = \uparrow* \quad \xrightarrow{*} \downarrow = \uparrow* \quad \xleftarrow{*} \downarrow = \uparrow* \quad \dots \quad (13)$$

is possible with the transition; the center case will obtain only then when both points should happen to coincide.

It is apparent from table V that point  $s_0$  passes from field VI towards I. We might be inclined to think that this takes place in a way corresponding to the transition from field VII towards VI; then at a certain moment of the osmosis point  $s_0$  would arrive on the side  $WX$ , so that at this moment no  $Y$  would pass through the membrane; this, however, is not possible, as we shall see directly.

If we represent the angles formed by the chords with the negative direction of the  $X$ -axis by  $\varphi$  and  $\varphi'$ , then, as we saw above, we have  $\operatorname{tg} \varphi = \mu$  and  $\operatorname{tg} \varphi' = \mu'$ . Consequently the directions of the chords follow from the values of  $\mu$  and  $\mu'$ . In table Vb we find these values for all chords of this system.

If we leave chord 1.2 (and 1'.2') of the path out of consideration, we see that the angles  $\varphi$  and  $\varphi'$  increase perpetually during the osmosis; this is also valid, therefore, for the angles  $\varphi$  and  $\varphi'$  which are formed by the tangents with the negative direction of the  $X$ -axis. We now see from table Vb that somewhere in the vicinity of point 6 a point  $u$  (and  $u'$ ) is situated, where  $\varphi = \varphi'$  and consequently the tangents run parallel. The point of intersection  $s_0$  of these tangents is situated now at infinite distance in field IV or VII; now we have the transition of  $s_0$  from field IV towards VII which has been discussed above.

In the vicinity of point  $q$ , however, is also situated a point  $v$  (and  $v'$ ) where the tangents become parallel again; now point  $s_0$  is situated again at infinite distance, but it now passes from field VI toward field III.

The analysis indicates, however, that point  $s_0$  of part 9.10 of the path is situated in field I; consequently point  $s_0$  has passed from field III towards I either through field II or through IV.

When the system passes along part 9.10 of its path it is situated

successively either in the fields VI, III, II and I or in VI, III, IV and I. The mixture  $s_0$ , experimentally determined, consists, therefore, of mixtures of very different fields.

This might perhaps be ascertained by having the system pass once more along part 9.10 of its path under precisely the same circumstances and by determining several points on this same part; but even, if this were possible, these points would be situated too close to one another for us to deduct from them the mixture  $s_0$  with sufficient accuracy.

Previously (Gen. V) we have seen already that the form of a path also depends on the quantity and composition of the liquid, contained by the membrane at every moment of the osmosis.

TABLE Vb.

	$\mu = \operatorname{tg} \varphi$	$\mu' = \operatorname{tg} \varphi'$
1.2	0.13469	0.15576
2.3	0.09678	0.13264
3.4	0.10009	0.14182
4.5	0.13163	0.16578
5.6	0.19456	0.19651
6.7	$u$ 0.29739	$u'$ 0.29252
7.8	0.44744	0.43070
8.9	$v$ 0.97485	$v'$ 0.72578
9.10	$v$ 1.0167	$v'$ 1.3558

If we take a membrane, as e.g. a pig's bladder or a piece of parchment that contains none of the diffusing substances, then this must first take in these substances before the osmosis can begin.

If we take a membrane, which already contains water, as e.g. collodion, then it still has to absorb the other substances, whereas it will perhaps give off water.

The form of the path, especially at the beginning of the osmosis, therefore, also depends more or less on the original condition of the membrane. Of course this change in form is no mistake of the system; it only indicates which changes have occurred on the left and on the right side of the membrane; it is the cause, however, that the mixture  $L_0$  in this first part of the path shows a greater deviation from the mixtures  $L_l$  and  $L_r$ .

It is apparent from table Vb that the angle  $\varphi$  is larger on part 1.2 of the path than for part 2.3 but afterwards increases all the time; the same is valid for the angle  $\varphi'$ . Therefore, at the beginning of the

osmosis both branches of the path have another form than at a later time. A similar phenomenon also occurs with several of the other paths.

Perhaps this is a result of the influence of the membrane at the beginning of the osmosis, as has been discussed above.

The path of system IV practically coincides with part V in fig. 1 Exp. I; it appears from the schemes that for system IV the same is valid as for V. Yet system IV shows a peculiarity which does not occur in V.

It namely appears from the table that point  $s_0$ , after having passed from field V towards VII, returns again to IV and then again passes towards VII; duploexperiments showed that mistakes in analysis did not play a part here. These leaps and bounds of the point  $s_0$ , however, have no influence on the directions in which the substances pass through the membrane, as is apparent from the scheme; of course they do influence the ratio of the quantities of these substances; in field IV less  $X$  goes towards the left than  $Y + W$  towards the right; in field VII the opposite thing is the case.

Perhaps this phenomenon can spring from different cases; we have to bear in mind here, that with these leaps and bounds the tangents run practically parallel. For the path now is not theoretical, but experimental; it consists of a succession of theoretical parts, which indeed pass into one another continuously, but yet do not belong to the same theoretical path. (Comp. Gen I); besides the temperature during the whole of the osmosis has not remained constant.

The changes, which the diffusing substances cause in the membrane during the osmosis, can be of more influence, however; this alteration can continuously change the permeability for one or more of the substances. We have had membranes [collodion + deposit  $Cu_2 Fe (CN)_6$ ] which had practically lost their permeability altogether after an osmosis of some days; after having been put into water some of them partially got it back again, others, however, did not.

For the systems II and III, the paths of which in fig. 1 Exp. I are situated between the paths I and V, the same is valid as for system V. In discussing this latter system we have seen that during its transition from field VI towards I point  $s_0$  is situated successively in the fields VI, III, II and I or in VI, III, IV and I. As point  $s_0$  of system II has been determined, however, also in field IV, it has here run through the fields VI, III, IV and I successively.

A study of the movement of point  $s_0$  in the other systems is left to the reader.

Previously (Gen. II) we have seen already that no direct correspondence exists between the direction in which one of the substances passes

through the membrane and the change in concentration of this substance on both sides of the membrane. Of this we find many examples in these systems. Let us take system VI e.g. We see that substance  $Y$  diffuses towards the right on the parts 1.2 and 2.3 of the path; yet the  $Y$ -amount of both liquids increases. On part 7.8 of the path the water diffuses towards the right; yet the  $W$ -amount still decreases on the right side and increases on the left side.

The left-side liquids  $L_1$  of the systems I—V only contain water +  $Na_2CO_3$ ; approximately they have the same composition. The right-side liquids  $L'_1$  contain water +  $NaCl$ ; they have very different compositions, however. Therefore, the five paths in fig. 1 Exp. I have on the  $Y$ -axis the same point 1 but on the  $X$ -axis a different point 1'. For the  $W$ -amount of part 1.2 of these paths is valid:



The first arrow obtains for system I, the second for the systems II—V. From this we learn the influence of the concentration of the right-side liquid on the direction, in which the water passes through the membrane.

The systems VI and VII (fig. 1. Exp. II) approximately have the same liquids on the left and the right side of the membrane. They have a different membrane, however. For the  $W$ -amount of part 1.2 of these paths we have:



For the systems XI and XII which also have approximately the same liquids on the left and the right side of the membrane (fig. 2 Exp. II) the same obtains.

Previously we have seen already that the membrane has a great influence on the duration of the osmosis; we see here that it can influence the very direction in which the water passes through the membrane.

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**Mathematics.** — *On the general theory of elastic stability.* By C. B. BIEZENO and H. HENCKY. (Communicated by Prof. W. VAN DER WOUDE).

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1. *Introduction.* Though the problem of stability of an elastic body has been solved for a great number of special cases, little attention has been given, up to now, to the general problem. It is true, that DUHEM<sup>1)</sup> in his "Recherches sur l'élasticité" develops a general stability-theory, but his deductions are so abstract, that practical results remain concealed. The first who in a concrete manner tackles the general problem is SOUTHWELL<sup>2)</sup>, but he restricts himself to the case, that the body, (before collapsing), finds itself in a state of homogeneous stress.

A paper of DEAN<sup>3)</sup> based on the above-mentioned one of SOUTHWELL, cannot be considered as a real generalisation of SOUTHWELL's results. A treatise of REISSNER<sup>4)</sup>, relating to ideas of BRYAN<sup>5)</sup>, confines itself to the flat plate.

In the following lines we therefore intend to deduce the differential-equations, which hold for the general case of elastic stability.

### I. *General equations of equilibrium.*

2. *Definitions.* In the following an elastic body may be considered, in equilibrium under the action of the volumetric forces  $X, Y, Z$  and the surface-tensions  $P_x, P_y, P_z$ . Then the stress-components  $S_x, S_y, S_z, T_x, T_y, T_z$  have to satisfy the equations:

$$\left. \begin{array}{l} \frac{\partial S_x}{\partial x} + \frac{\partial T_z}{\partial y} + \frac{\partial T_y}{\partial z} + X = 0 \\ \frac{\partial T_z}{\partial x} + \frac{\partial S_y}{\partial y} + \frac{\partial T_x}{\partial z} + Y = 0 \\ \frac{\partial T_y}{\partial x} + \frac{\partial T_x}{\partial y} + \frac{\partial S_z}{\partial z} + Z = 0 \end{array} \right\} \quad \dots \quad (1)$$

<sup>1)</sup> *Recherches sur l'élasticité.* Paris 1906, p. 83—132.

<sup>2)</sup> *Phil. Transactions Roy. soc.* 1913 ser. A. Vol. 213, p. 187—244.

<sup>3)</sup> *Proc. of the Roy. soc. of London* ser. A. Vol. 107, 1925. p. 734—760.

<sup>4)</sup> *Z. A. M. M. Bd. 5,* 1925, p. 475—478.

<sup>5)</sup> *Proc. of the Cambridge phil. soc.* Vol. 6, 1887, p. 199—210 and p. 287—292.

*Proc. of the London math. soc.* Vol. 22, 1891, p. 54—67.

at every internal point of the body, and the boundary-conditions

$$\left. \begin{aligned} P_x &= S_x \cdot \cos(nx) + T_z \cdot \cos(ny) + T_y \cdot \cos(nz) \\ P_y &= T_z \cdot \cos(nx) + S_y \cdot \cos(ny) + T_x \cdot \cos(nz) \\ P_z &= T_y \cdot \cos(nx) + T_x \cdot \cos(ny) + S_z \cdot \cos(nz) \end{aligned} \right\} \quad \dots \quad (2)$$

( $n$  = outward directed normal of the boundary surface) at every point of the boundary-surface.

The state of strain, induced by the stress system  $S_x, S_y, S_z, T_x, T_y, T_z$ , indicated by I, will be considered, as the zero-state, from where displacements  $u, v, w$  as yet to be introduced will be measured. In state I every point of the body therefore has displacements  $u, v, w = 0$ .

Now another state of strain, indicated by II, defined by the displacements  $u, v, w$  of the different points of the body is introduced, and the question is put forward, which stresses must be added on the faces of a small rectangular parallelepiped  $dx, dy, dz$ , cut from the body in state I, and subjected to the surface forces  $S_x dy \cdot dz \dots$ , to bring it into the state II.

Under the action of the additional forces the element must be brought in a new shape and a new position, each of which are entirely determined by the displacements  $u, v, w$  and their derivations with respect to  $x, y, z$ . The change in shape is given by the quantities :

$$\frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial w}{\partial z}, f_x = \frac{1}{2} \left( \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right), f_y = \frac{1}{2} \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right), f_z = \frac{1}{2} \left( \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right), \quad (3a)$$

the change in position by the translation-components  $u, v, w$  (which are inessential for our problem) and the rotation-components

$$\omega_x = \frac{1}{2} \left( \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right), \quad \omega_y = \frac{1}{2} \left( \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right), \quad \omega_z = \frac{1}{2} \left( \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right). \quad (3b)$$

To determine the additional stresses just mentioned, the element under consideration, subjected in state II to the action of the volumetric forces and the surface-tensions is rotated with those forces over the angles  $-\omega_x, -\omega_y, -\omega_z$  about the axes  $x, y, z$ , so as to assume a new position indicated by II'. Comparison of the two states I and II' then informs us which additional surface-forces must be introduced in state I to give the element its proper shape of state II', and therefore its shape of state II. These additional forces divided by the corresponding surfaces of the element in state I, define the additional stresses. These stresses cannot be the components of a symmetrical stress-tensor, as would have been the case when the element in state I had been free from forces. For, if the element in state I is subjected to the action of additional forces, the points of application of the surface-forces already present,

(to be called  $A$  and  $A'$  for the faces  $\perp$  the  $x$ -axis,  $B$  and  $B'$  for those  $\perp$  the  $y$ -axis and  $C, C'$  for those  $\perp$  the  $z$ -axis) will tend to a new configuration. In consequence the forces, already present, give rise to moments which must be neutralised by the additional forces. Therefore these additional forces have to be characterised by the following scheme of stress-components:

$$\begin{array}{lll} s_x & t_x + r_z & t_y - r_y \\ t_x - r_z & s_y & t_x + r_x \\ t_y + r_y & t_x - r_x & s_z \end{array}$$

in which the three quantities of the first row, multiplied by  $dy dz$ , indicate the force-components, to be added on the faces originally loaded by  $S_x, dy dz, T_z, dy dz, T_y, dy dz$ , a. s. o. .

It must be noticed that only the stresses  $s_x, s_y, s_z, t_x, t_y, t_z$  are responsible for the change in shape of the element, and that in consequence only these components have to be relied with the  $u, v, w$  by stress-strain equations.

The quantities  $r_x, r_y, r_z$  are determined by equations of equilibrium against rotation.

3. To obtain these equations, we have to examine the configuration  $AA', BB', CC'$  in state II'. As can easily be seen the components of  $AA', BB', CC'$  in that state are given by the scheme:

$$\begin{array}{lll} x & y & z \\ AA': dx \left( 1 + \frac{\partial u}{\partial x} \right), & dx \left( \frac{\partial v}{\partial x} - \omega_z \right) = dx \cdot f_z, & dx \left( \frac{\partial w}{\partial x} + \omega_y \right) = dx \cdot f_y, \\ BB': dy \left( \frac{\partial u}{\partial y} + \omega_z \right) = dy \cdot f_z, & dy \left( 1 + \frac{\partial v}{\partial y} \right), & dy \left( \frac{\partial w}{\partial y} - \omega_x \right) = dy \cdot f_x, \\ CC': dz \left( \frac{\partial u}{\partial z} - \omega_y \right) = dz \cdot f_y, & dz \left( \frac{\partial v}{\partial z} + \omega_x \right) = dz \cdot f_x, & dz \left( 1 + \frac{\partial w}{\partial z} \right). \end{array}$$

The components of the forces, acting in  $A$  and  $A'$  have the magnitude:

$$(S_x + s_x) dy dz, (T_z + t_z + r_z) dy dz, (T_y + t_y - r_y) dy dz .$$

those of the forces acting in  $B$  and  $B'$ :

$$(T_z + t_z - r_z) dz dx, (S_y + s_y) dz dx, (T_z + t_z + r_z) dz dx$$

those of the forces acting in  $C$  and  $C'$ :

$$(T_y + t_y + r_y) dx dy, (T_x + t_x - r_x) dx dy, (S_z + s_z) dx dy.$$

The moment of all these forces about the axis of  $x$  must be zero, so that:

$$(T_y + t_y - r_y) f_z - (T_z + t_z + r_z) f_y + (T_x + t_x + r_x) \left( 1 + \frac{\partial v}{\partial y} \right) - (S_y + s_y) f_x \\ + (S_z + s_z) f_x - (T_x + t_x - r_x) \left( 1 + \frac{\partial w}{\partial z} \right) = 0.$$

Neglecting terms, quadratic in the derivatives of  $u, v, w$  and in  $s_x, \dots, t_x, \dots, r_x, \dots$ , we obtain

$$2 \cdot r_x - (S_y - S_z) f_x - T_z f_y + T_y f_z - T_x \left( \frac{\partial w}{\partial z} - \frac{\partial v}{\partial y} \right) = 0.$$

The three equations, determining the quantities  $r_x, r_y, r_z$  therefore are:

$$\left. \begin{aligned} 2 \cdot r_x &= (S_y - S_z) f_x - T_z f_y - T_y f_z + T_x \left( \frac{\partial w}{\partial z} - \frac{\partial v}{\partial y} \right) \\ 2 \cdot r_y &= (S_z - S_x) f_y + T_x f_z - T_z f_x + T_y \left( \frac{\partial u}{\partial x} - \frac{\partial w}{\partial z} \right) \\ 2 \cdot r_z &= (S_x - S_y) f_z + T_y f_x - T_x f_y + T_z \left( \frac{\partial v}{\partial y} - \frac{\partial u}{\partial x} \right) \end{aligned} \right\} \quad \dots \quad (4)$$

4. The equations expressing the equilibrium of the element in the directions  $x, y, z$  will be obtained by the examination of state II.

Paying attention to the face, the centre of which was indicated by  $A$ , we had, as components of the force acting in the directions  $x, y, z$ , in state II':

$$(S_x + s_x) dy dz, \quad (T_z + t_z + r_z) dy dz, \quad (T_y + t_y - r_y) dy dz.$$

By rotating these forces over the angles  $(\omega_x, \omega_y, \omega_z)$  about the axes  $x, y, z$ , we obtain the three components of the force acting in  $A$  in state II, however in directions differing from  $x, y, z$ .

It is easily understood, that the components of the same force in those latter directions can be written as follows:

$$\left. \begin{aligned} X_x &= S_x + s_x + T_y \omega_y - T_z \omega_z \\ Y_x &= T_z + t_z + r_z + S_x \omega_z - T_y \omega_x \\ Z_x &= T_y + t_y - r_y + T_z \omega_x - S_x \omega_y \end{aligned} \right\} \quad \dots \quad (5^a)^1)$$

<sup>1)</sup> It must be emphasized that  $X_x, Y_x, Z_x$  are the stress-components in the directions  $x, y, z$ , on an element, which originally was perpendicular to the axis  $x$ , and that  $X_y, Y_y, Z_y$ , resp.  $X_z, Y_z, Z_z$  have a corresponding meaning.

when terms, quadratic in  $s_x, \dots, t_x, \dots, r_x, \dots$  are neglected. In a similar manner we find:

$$\left. \begin{aligned} X_y &= T_z + t_z - r_z + T_x \omega_y - S_y \omega_z \\ Y_y &= S_y + s_y + T_z \omega_z - T_x \omega_x \\ Z_y &= T_x + t_x + r_x + S_y \omega_x - T_z \omega_y \end{aligned} \right\} \dots \dots \quad (5^b)$$

resp.

$$\left. \begin{aligned} X_z &= T_y + t_y + r_y + S_z \omega_y - T_x \omega_z \\ Y_z &= T_x + t_x - r_x + T_y \omega_z - S_z \omega_x \\ Z_z &= S_z + s_z + T_x \omega_x - T_y \omega_y \end{aligned} \right\} \dots \dots \quad (5^c)$$

The ordinary equations of equilibrium

$$\begin{aligned} \frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z} + X &= 0 \\ \frac{\partial Y_x}{\partial x} + \frac{\partial Y_y}{\partial y} + \frac{\partial Y_z}{\partial z} + Y &= 0 \\ \frac{\partial Z_x}{\partial x} + \frac{\partial Z_y}{\partial y} + \frac{\partial Z_z}{\partial z} + Z &= 0 \end{aligned}$$

therefore take the form

$$\left. \begin{aligned} L &\equiv \frac{\partial s_x}{\partial x} + \frac{\partial t_x}{\partial y} + \frac{\partial r_x}{\partial z} + T_y \cdot \frac{\partial \omega_y}{\partial x} + T_x \cdot \frac{\partial \omega_y}{\partial y} + S_z \cdot \frac{\partial \omega_y}{\partial z} \\ &\quad - T_z \cdot \frac{\partial \omega_x}{\partial x} - S_y \cdot \frac{\partial \omega_x}{\partial y} - T_x \cdot \frac{\partial \omega_x}{\partial z} \\ &\quad - \frac{\partial r_z}{\partial y} + \frac{\partial r_y}{\partial z} + Y \cdot \omega_z - Z \cdot \omega_y = 0 \end{aligned} \right\}. \quad (6^a)$$

$$\left. \begin{aligned} M &\equiv \frac{\partial t_x}{\partial x} + \frac{\partial s_y}{\partial y} + \frac{\partial r_x}{\partial z} + S_x \cdot \frac{\partial \omega_z}{\partial x} + T_z \cdot \frac{\partial \omega_z}{\partial y} + T_y \cdot \frac{\partial \omega_z}{\partial z} \\ &\quad - T_y \cdot \frac{\partial \omega_x}{\partial x} - T_x \cdot \frac{\partial \omega_x}{\partial y} - S_z \cdot \frac{\partial \omega_x}{\partial z} \\ &\quad - \frac{\partial r_x}{\partial z} + \frac{\partial r_z}{\partial x} + Z \cdot \omega_x - X \cdot \omega_z = 0 \end{aligned} \right\}. \quad (6^b)$$

$$\left. \begin{aligned} N &\equiv \frac{\partial t_y}{\partial x} + \frac{\partial s_z}{\partial y} + \frac{\partial r_y}{\partial z} + T_z \cdot \frac{\partial \omega_x}{\partial x} + S_y \cdot \frac{\partial \omega_x}{\partial y} + T_x \cdot \frac{\partial \omega_x}{\partial z} \\ &\quad - S_x \cdot \frac{\partial \omega_y}{\partial x} - T_z \cdot \frac{\partial \omega_y}{\partial y} - T_y \cdot \frac{\partial \omega_y}{\partial z} \\ &\quad - \frac{\partial r_y}{\partial x} + \frac{\partial r_x}{\partial y} + X \cdot \omega_y - Y \cdot \omega_x = 0 \end{aligned} \right\}. \quad (6^c)$$

5. Now we have to introduce the stress-strain relations. As stated before, only the quantities  $s_x, s_y, s_z, t_x, t_y, t_z$  affect the change of shape of the element; therefore we may write:

$$\left. \begin{aligned} s_x &= \frac{2G}{m-2} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) + 2G \cdot \frac{\partial u}{\partial x} \\ s_y &= \frac{2G}{m-2} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) + 2G \cdot \frac{\partial v}{\partial y} \\ s_z &= \frac{2G}{m-2} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) + 2G \cdot \frac{\partial w}{\partial z} \\ t_x &= \dots \quad + 2G \cdot f_x \\ t_y &= \dots \quad + 2G \cdot f_y \\ t_z &= \dots \quad + 2G \cdot f_z \end{aligned} \right\} \text{1) . . . (7)}$$

The differential equations for  $u, v, w$  are now obtained by substituting the values of  $r_x, r_y, r_z$  from (4) in (6a, b and c) and by substituting then the expressions (7) in these results. In the special case that  $S_x, \dots, T_x$  are zero, these equations reduce to the wellknown fundamental equations:

$$\begin{aligned} \Delta u + \frac{m}{m-2} \cdot \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) &= 0 \\ \Delta v + \frac{m}{m+2} \cdot \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) &= 0 \\ \Delta w + \frac{m}{m-2} \cdot \frac{\partial}{\partial z} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) &= 0 \end{aligned}$$

6. The foregoing deductions have general validity and do not restrict themselves to questions of stability. The characteristic feature of the stability problem is furnished by the boundary conditions, which involve that to a solution  $u, v, w$  another solution  $\lambda u, \lambda v, \lambda w$  ( $\lambda$  sufficiently small) can be obtained.

To establish those boundary conditions we consider a boundary tetrahedron in state II, as well as in the intermediate state II'. During the transition from II' to II all forces rotate over the local angle of rotation  $\omega$ .

If  $dO_1$  ( $= \frac{1}{2} dy dz$ ) be the surface in state I of that tetrahedron-face,

1) These relations may be called in question, and indeed require further investigation, which the authors reserve to themselves; on the other side their usefulness is proved by the results obtained in special cases, to be mentioned in chapter III.

which originally was perpendicular to the  $x$ -axis, then the forces on that same face in state II acting again in the  $x$ -direction are:

$$(S_x + s_x) dO_1 - (T_z + t_z + r_z) \omega_z dO_1 + (T_y + t_y - r_y) \omega_y dO_1$$

Similarly the  $x$ -components of the forces acting on the faces, originally perpendicular to the axes  $y$  and  $z$ , have the magnitude:

$$(T_z + t_z - r_z) dO_2 - (S_y + s_y) \omega_z dO_2 + (T_x + t_x + r_x) \omega_y dO_2$$

resp.

$$(T_y + t_y + r_y) dO_3 - (T_x + t_x - r_x) \omega_z dO_3 + (S_z + s_z) \omega_y dO_3.$$

If the boundary surface is subjected to elastic forces, depending linearly on  $u, v, w$ , then we have to introduce on the boundary face of the tetrahedron, besides the forces  $P_x, dO, P_y dO, P_z, dO$  other ones, given by

$$\begin{aligned} & - (k_1 u + l_1 v + m_1 w) dO \\ & - (k_2 u + l_2 v + m_2 w) dO \\ & - (k_3 u + l_3 v + m_3 w) dO. \end{aligned}$$

Now the equations of equilibrium of the tetrahedron can be written down, and we find, (with regard to the equations (2)):

$$\left. \begin{aligned} & s_x \cos(n x) + (t_z - r_z) \cos(n y) + (t_y + r_y) \cos(n z) + \\ & + P_z \omega_y - P_y \omega_z + k_1 u + l_1 v + m_1 w = 0 \\ & (t_z + r_z) \cos(n x) + s_y \cos(n y) + (t_x - r_x) \cos(n z) + \\ & + P_x \omega_z - P_z \omega_x + k_2 u + l_2 v + m_2 w = 0 \\ & (t_y - r_y) \cos(n x) + (t_x + r_x) \cos(n y) + s_z \cos(n z) + \\ & + P_y \omega_x - P_x \omega_y + k_3 u + l_3 v + m_3 w = 0 \end{aligned} \right\} \quad (8)$$

From (6) and (8) we derive, that if a solution  $u, v, w$  exists, differing from zero, another solution  $\lambda u, \lambda v, \lambda w$  is admissible too.

## II. Second derivation of the general equations of equilibrium.

7. Another way, in which the equations (6), and (8) can be obtained is the following:

Suppose that the displacements  $u, v, w$  require the boundary stresses  $P_x + \Delta P_x, P_y + \Delta P_y, P_z + \Delta P_z$  to sustain the equilibrium in state II.

Then this equilibrium requires, that for any set of virtual displacements  $\delta u, \delta v, \delta w$ :

$$\left. \begin{aligned}
 & \iint dO \{ (P_x + \Delta P_x) \delta u + (P_y + \Delta P_y) \delta v + (P_z + \Delta P_z) \delta w \} + \\
 & \quad + \iiint dV (X \delta u + Y \delta v + Z \delta w) \\
 & - \iiint dV \left( X_x \cdot \frac{\partial \delta u}{\partial x} + X_y \cdot \frac{\partial \delta u}{\partial y} + X_z \cdot \frac{\partial \delta u}{\partial z} + \right. \\
 & \quad \left. Y_x \cdot \frac{\partial \delta v}{\partial x} + Y_y \cdot \frac{\partial \delta v}{\partial y} + Y_z \cdot \frac{\partial \delta v}{\partial z} + \right. \\
 & \quad \left. Z_x \cdot \frac{\partial \delta w}{\partial x} + Z_y \cdot \frac{\partial \delta w}{\partial y} + Z_z \cdot \frac{\partial \delta w}{\partial z} \right) = 0
 \end{aligned} \right\}. \quad (9)$$

The stress-components  $X_{x\dots}$  are defined by the equations (5a, b, c);  $dO$  and  $dV$  both refer to state I, in accordance to what has been stated in n°. 2. Transformation of the second volume-integral in equation (9) leads to

$$\left. \begin{aligned}
 & \iint dO \{ [P_x + \Delta P_x - X_x \cos(nx) - X_y \cos(ny) - X_z \cos(nz)] \delta u \\
 & \quad + [P_y + \Delta P_y - Y_x \cos(nx) - Y_y \cos(ny) - Y_z \cos(nz)] \delta v \\
 & \quad + [P_z + \Delta P_z - Z_x \cos(nx) - Z_y \cos(ny) - Z_z \cos(nz)] \delta w \} \\
 & + \iiint dV \left\{ \left( \frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z} + X \right) \delta u + \right. \\
 & \quad \left. + \left( \frac{\partial Y_x}{\partial x} + \frac{\partial Y_y}{\partial y} + \frac{\partial Y_z}{\partial z} + Y \right) \delta v + \left( \frac{\partial Z_x}{\partial x} + \frac{\partial Z_y}{\partial y} + \frac{\partial Z_z}{\partial z} + Z \right) \delta w \right\} = 0
 \end{aligned} \right\} \quad (10)$$

By appropriate specialisation of the displacements  $\delta u, \delta v, \delta w$  the equations (6a, b, c) as well as the equations (8) can be obtained.

By putting  $\delta u = \delta v = \delta w = 0 \dots$  at all points of the boundary surface, equation (10), by means of the equations of equilibrium for state I, can be simplified to:

$$\iiint dV (L \delta u + M \delta v + N \delta w) = 0 \dots \quad (11)$$

which represents the three equations

$$L = 0 \quad M = 0 \quad N = 0.$$

In consequence of these equations the surface-integral occurring in (10), must be zero too. Combining the equations with the boundary-conditions for state I, we find the three relations,

$$\left. \begin{aligned}
 \Delta P_x &= s_x \cos(nx) + (t_z - r_z) \cos(ny) + (t_y + r_y) \cos(nz) + P_z \omega_y - P_y \omega_z \\
 \Delta P_y &= (t_z + r_z) \cos(nx) + s_y \cos(ny) + (t_x - r_x) \cos(nz) + P_x \omega_z - P_z \omega_x \\
 \Delta P_z &= (t_y - r_y) \cos(nx) + (t_x + r_x) \cos(ny) + s_z \cos(nz) + P_y \omega_x - P_x \omega_y
 \end{aligned} \right\} \quad (12)$$

yielding the amounts of  $\Delta P_x, \Delta P_y, \Delta P_z, \dots$ , necessary to sustain the equilibrium of state II. The equations (12) become identical with the equations (8), when the problem of stability is characterised by the conditions

$$\Delta P_x = -(k_1 u + l_1 v + m_1 w)$$

$$\Delta P_y = -(k_2 u + l_2 v + m_2 w)$$

$$\Delta P_z = -(k_3 u + l_3 v + m_3 w)$$

8. In a similar manner the equations (4) can be derived from (9), when this equation is applied to a volume-element, under the restriction, that only those variations  $\delta u, \delta v, \delta w$  are admissible, by which the element moves like a rigid body. In that case

$$\delta f_x = \delta f_y = \delta f_z = \delta \left( \frac{\partial u}{\partial x} \right) = \delta \left( \frac{\partial v}{\partial y} \right) = \delta \left( \frac{\partial w}{\partial z} \right) = 0$$

and (9) reduces to:

$$\left. \begin{aligned} X_x \cdot \frac{\partial \delta u}{\partial x} + X_y \cdot \frac{\partial \delta u}{\partial y} + X_z \cdot \frac{\partial \delta u}{\partial z} + Y_x \cdot \frac{\partial \delta v}{\partial x} + Y_y \cdot \frac{\partial \delta v}{\partial y} + Y_z \cdot \frac{\partial \delta v}{\partial z} \\ + Z_x \cdot \frac{\partial \delta w}{\partial x} + Z_y \cdot \frac{\partial \delta w}{\partial y} + Z_z \cdot \frac{\partial \delta w}{\partial z} = 0 \end{aligned} \right\} \quad (13)$$

Attention must be drawn to the fact, that  $\frac{\partial \delta u}{\partial x} \dots$  represents a local derivative, and that  $\delta \left( \frac{\partial u}{\partial x} \right), \dots$  are variations of the straincomponents of determinate material particle so that  $\frac{\partial \delta u}{\partial x}$  and  $\delta \left( \frac{\partial u}{\partial x} \right)$  are unequal quantities. The relations between these quantities can be found by identifying the transformation:

$$(dx_3, dy_3, dz_3) = \begin{vmatrix} 1 + \frac{\partial (u + \delta u)}{\partial x}, & \frac{\partial (u + \delta u)}{\partial y}, & \frac{\partial (u + \delta u)}{\partial z} \\ \frac{\partial (v + \delta v)}{\partial x}, 1 + \frac{\partial (v + \delta v)}{\partial y}, & \frac{\partial (v + \delta v)}{\partial z} \\ \frac{\partial (w + \delta w)}{\partial x}, & \frac{\partial (w + \delta w)}{\partial y}, 1 + \frac{\partial (w + \delta w)}{\partial z} \end{vmatrix} (dx, dy, dz)$$

and the resultant of the two transformations

$$(dx_2, dy_2, dz_2) = \begin{vmatrix} 1 + \frac{\partial u}{\partial x}, & \frac{\partial u}{\partial y}, & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x}, 1 + \frac{\partial v}{\partial y}, & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x}, & \frac{\partial w}{\partial y}, 1 + \frac{\partial w}{\partial z} \end{vmatrix} (dx, dy, dz)$$

$$(dx_3, dy_3, dz_3) = \begin{vmatrix} 1 + \delta \left( \frac{\partial u}{\partial x} \right), & \delta \left( \frac{\partial u}{\partial y} \right), & \delta \left( \frac{\partial u}{\partial z} \right) \\ \delta \left( \frac{\partial v}{\partial x} \right), & 1 + \delta \left( \frac{\partial v}{\partial y} \right), & \delta \left( \frac{\partial v}{\partial z} \right) \\ \delta \left( \frac{\partial w}{\partial x} \right), & \delta \left( \frac{\partial w}{\partial y} \right), & 1 + \delta \left( \frac{\partial w}{\partial z} \right) \end{vmatrix} (dx_2, dy_2, dz_2)$$

In this way we find:

$$\left. \begin{aligned} \frac{\partial \delta u}{\partial x} &= \delta \left( \frac{\partial u}{\partial x} \right) + \frac{\partial u}{\partial x} \cdot \delta \left( \frac{\partial u}{\partial x} \right) + \frac{\partial v}{\partial x} \cdot \delta \left( \frac{\partial u}{\partial y} \right) + \frac{\partial w}{\partial x} \cdot \delta \left( \frac{\partial u}{\partial z} \right) \\ \frac{\partial \delta u}{\partial y} &= \delta \left( \frac{\partial u}{\partial y} \right) + \frac{\partial u}{\partial y} \cdot \delta \left( \frac{\partial u}{\partial x} \right) + \frac{\partial v}{\partial y} \cdot \delta \left( \frac{\partial u}{\partial y} \right) + \frac{\partial w}{\partial y} \cdot \delta \left( \frac{\partial u}{\partial z} \right) \\ \frac{\partial \delta u}{\partial z} &= \delta \left( \frac{\partial u}{\partial z} \right) + \frac{\partial u}{\partial z} \cdot \delta \left( \frac{\partial u}{\partial x} \right) + \frac{\partial v}{\partial z} \cdot \delta \left( \frac{\partial u}{\partial y} \right) + \frac{\partial w}{\partial z} \cdot \delta \left( \frac{\partial u}{\partial z} \right) \\ \frac{\partial \delta v}{\partial x} &= \delta \left( \frac{\partial v}{\partial x} \right) + \frac{\partial u}{\partial x} \cdot \delta \left( \frac{\partial v}{\partial x} \right) + \frac{\partial v}{\partial x} \cdot \delta \left( \frac{\partial v}{\partial y} \right) + \frac{\partial w}{\partial x} \cdot \delta \left( \frac{\partial v}{\partial z} \right) \\ \frac{\partial \delta v}{\partial y} &= \delta \left( \frac{\partial v}{\partial y} \right) + \frac{\partial u}{\partial y} \cdot \delta \left( \frac{\partial v}{\partial x} \right) + \frac{\partial v}{\partial y} \cdot \delta \left( \frac{\partial v}{\partial y} \right) + \frac{\partial w}{\partial y} \cdot \delta \left( \frac{\partial v}{\partial z} \right) \\ \frac{\partial \delta v}{\partial z} &= \delta \left( \frac{\partial v}{\partial z} \right) + \frac{\partial u}{\partial z} \cdot \delta \left( \frac{\partial v}{\partial x} \right) + \frac{\partial v}{\partial z} \cdot \delta \left( \frac{\partial v}{\partial y} \right) + \frac{\partial w}{\partial z} \cdot \delta \left( \frac{\partial v}{\partial z} \right) \\ \frac{\partial \delta w}{\partial x} &= \delta \left( \frac{\partial w}{\partial x} \right) + \frac{\partial u}{\partial x} \cdot \delta \left( \frac{\partial w}{\partial x} \right) + \frac{\partial v}{\partial x} \cdot \delta \left( \frac{\partial w}{\partial y} \right) + \frac{\partial w}{\partial x} \cdot \delta \left( \frac{\partial w}{\partial z} \right) \\ \frac{\partial \delta w}{\partial y} &= \delta \left( \frac{\partial w}{\partial y} \right) + \frac{\partial u}{\partial y} \cdot \delta \left( \frac{\partial w}{\partial x} \right) + \frac{\partial v}{\partial y} \cdot \delta \left( \frac{\partial w}{\partial y} \right) + \frac{\partial w}{\partial y} \cdot \delta \left( \frac{\partial w}{\partial z} \right) \\ \frac{\partial \delta w}{\partial z} &= \delta \left( \frac{\partial w}{\partial z} \right) + \frac{\partial u}{\partial z} \cdot \delta \left( \frac{\partial w}{\partial x} \right) + \frac{\partial v}{\partial z} \cdot \delta \left( \frac{\partial w}{\partial y} \right) + \frac{\partial w}{\partial z} \cdot \delta \left( \frac{\partial w}{\partial z} \right) \end{aligned} \right\} . \quad (14)$$

By substituting the first members of these equations in (13) we find, by neglecting terms of the second order in  $\frac{\partial u}{\partial x}, \dots, s_x, \dots, r_z$ ,

$$\begin{aligned} \delta \omega_x \left( + 2 \cdot r_x - T_z \frac{\partial w}{\partial x} - S_y \frac{\partial w}{\partial y} - T_x \frac{\partial w}{\partial z} - T_y \omega_z + S_z \cdot \omega_x + \right. \\ \left. + T_y \frac{\partial v}{\partial x} + T_z \frac{\partial v}{\partial y} + S_z \frac{\partial v}{\partial z} + S_y \omega_x - T_x \omega_y \right) + \\ + \delta \omega_y (\dots \text{cycl.}) + \delta \omega_z (\dots \text{cycl.}) = 0 \end{aligned}$$

which is equivalent to the system (4).

**Mathematics.** — *On the general theory of elastic stability.* (Second communication). By C. B. BIEZENO and H. HENCKY. (Communicated by Prof. W. VAN DER WOUDE).

(Communicated at the meeting of January 28, 1928.)

### III. *Applications.*

#### 9. *Introduction.*

In the following paragraphs we give some applications of the general equations, deduced in a previous paper (see vol. 31, p. 569). Already in the simplest cases great difficulties arise when the problem is treated in an exact manner, and therefore we shall have to look after admissible simplifications. The most evident one is to restrict by a suitable "kinematic scheme" the mobility of the different points of the body. By such a kinematic scheme, not only the displacements  $u, v, w$  but also their variations  $\delta u, \delta v, \delta w$  are affected, so that the integral:

$$\iiint dV (L \delta u + M \delta v + N \delta w) = 0$$

which is to be looked upon as the representant of the equations (6a, b, c) may increase in usefulness. In the following examples this integral will always be our starting point.

#### 10. *The flat plate, supported elastically and subjected to volumetric forces.*

We suppose the plate lying in a vertical plane, the  $z$ -axis being directed normally to this plane. A deflection  $w$  in the direction of the axis of  $z$  involves a local elastic reaction of the magnitude  $-kw$ . Except to volumetric-forces the plate is only subjected to edge-stresses.

Both edge-stresses and volumetric forces are supposed to act in the  $xy$  plane.

The kinematic scheme is characterised by the statement:

1. that points, originally lying on a normal of the plate surface, after deformation find themselves again on a normal of the deformed middle-plane of the plate,
2. that strain in the direction of  $z$  will be excluded;
3. that for points of the plane  $z=0$ ,  $u$  and  $v$  will be zero.

Consequently the displacements of any point  $(x, y, z)$  must be put equal to :

$$\left. \begin{array}{l} u = -\frac{\partial w}{\partial x} \cdot z \\ v = -\frac{\partial w}{\partial y} \cdot z \\ w = w \end{array} \right\} \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (16)$$

where  $w$ , indicating the displacement of a point in the plane  $z=0$ , is a function of  $x$  and  $y$  only.

By the relations (16) we conclude from 3<sup>a</sup> and 3<sup>b</sup>:

$$\left. \begin{array}{l} \omega_x = +\frac{\partial w}{\partial y} \quad f_x = 0 \\ \omega_y = -\frac{\partial w}{\partial x} \quad f_y = 0 \\ \omega_z = 0 \quad f_z = -\frac{\partial^2 w}{\partial x \partial y} \cdot z \end{array} \right\} \quad \dots \quad \dots \quad \dots \quad \dots \quad (16^a)$$

and from (4)

$$\left. \begin{array}{l} 2 \cdot r_x = -T_y f_z - T_z \frac{\partial v}{\partial y} \\ 2 \cdot r_y = +T_x f_z + T_z \frac{\partial u}{\partial x} \\ 2 \cdot r_z = (S_x - S_y) f_z + T_z \left( \frac{\partial v}{\partial y} - \frac{\partial u}{\partial x} \right) \end{array} \right\} \quad \dots \quad \dots \quad \dots \quad \dots \quad (16^b)$$

Substituting the variations

$$\left. \begin{array}{l} \delta u = -\frac{\partial \delta w}{\partial x} \cdot z \\ \delta v = -\frac{\partial \delta w}{\partial y} \cdot z \\ \delta w = \delta w \end{array} \right\} \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (16^c)$$

in equation (11), we find :

$$\iiint dx \cdot dy \cdot dz \left\{ L \left( -\frac{\partial \delta w}{\partial x} \cdot z \right) + M \left( -\frac{\partial \delta w}{\partial y} \cdot z \right) + N \delta w \right\} = 0$$

which may be transformed in :

$$\begin{aligned} \iint dx \cdot dy \left\{ \int N dz + \int z \cdot \frac{\partial L}{\partial x} \cdot dz + \int z \cdot \frac{\partial M}{\partial y} \cdot dz \right\} \delta w \\ - \int ds [\cos(nx) \int L z \, dz + \cos(ny) \int M z \, dz] \delta w = 0. \end{aligned}$$

This equation holds for all possible variations of  $w$  and for every part of the plate. Putting  $\delta w = 0$  for all edge-points, we find:

$$\iint dx \cdot dy \cdot \delta w \left( N + z \cdot \frac{\partial L}{\partial x} + z \cdot \frac{\partial M}{\partial y} \right) dz = 0.$$

Therefore

$$\iint \left( N + z \cdot \frac{\partial L}{\partial x} + z \cdot \frac{\partial M}{\partial y} \right) dz = 0$$

and in consequence

$$\int ds \cdot \delta w \left[ \cos(nx) \int Lz dz + \cos(ny) \int Mz dz \right] = 0.$$

Application of the same reasoning to a part of the plate with contour  $C$ , leads to:

$$\int ds \cdot \delta w \left[ \cos(nx) \int Lz dz + \cos(ny) \int Mz dz \right] = 0$$

so that for every point  $Q$  of this contour

$$\cos(nx) \int Lz dz + \cos(ny) \int Mz dz = 0.$$

As for such a point  $Q$  the factors  $\cos(nx)$  and  $\cos(ny)$  vary with varying contour  $C$ , it follows, that for every point of the plane  $z = 0$ :

$$\text{I. } \int Lz dz = 0$$

$$\text{II. } \int Mz dz = 0$$

and consequently

$$\int z \cdot \frac{\partial L}{\partial x} \cdot dz = 0 \quad \int z \cdot \frac{\partial M}{\partial y} \cdot dz = 0.$$

From  $\int \left( N + z \cdot \frac{\partial L}{\partial x} + z \cdot \frac{\partial M}{\partial y} \right) dz = 0$  it follows that:

$$\text{III. } \int N dz = 0.$$

11. Restricting ourselves to the case  $T_x = T_y = S_z = Z = 0$  and  $S_x, S_y$  and  $T_z$  independent of  $z$ , we find:

$$L = \frac{\partial s_x}{\partial x} + \frac{\partial t_z}{\partial y} + \frac{\partial t_y}{\partial z} - \frac{\partial r_z}{\partial y}$$

$$M = \frac{\partial t_z}{\partial x} + \frac{\partial s_y}{\partial y} + \frac{\partial t_x}{\partial z} + \frac{\partial r_x}{\partial x}$$

$$N = \frac{\partial t_y}{\partial x} + \frac{\partial t_x}{\partial y} + \frac{\partial s_z}{\partial z} + T_z \cdot \frac{\partial \omega_x}{\partial x} + S_y \cdot \frac{\partial \omega_x}{\partial y} - S_x \cdot \frac{\partial \omega_y}{\partial x} - T_z \cdot \frac{\partial \omega_y}{\partial y} + X \omega_y - Y \omega_x.$$

Substituting these expressions in I and II, and integrating with respect to  $z$ , we find, with the abbreviations

$$M_x = \int s_x \cdot z \cdot dz$$

$$M_y = \int s_y \cdot z \cdot dz$$

$$D = \int t_z \cdot z \cdot dz:$$

$$\int t_y \cdot dz = \frac{\partial M_x}{\partial x} + \frac{\partial D}{\partial y} - \int \frac{\partial t_z}{\partial y} \cdot z \cdot dz \quad \dots \quad \dots \quad \dots \quad (I')$$

$$\int t_x \cdot dz = \frac{\partial D}{\partial x} + \frac{\partial M_y}{\partial y} + \int \frac{\partial t_z}{\partial x} \cdot z \cdot dz \quad \dots \quad \dots \quad \dots \quad (II')$$

from which the following equation can be deduced:

$$\int \frac{\partial t_y}{\partial x} \cdot dz + \int \frac{\partial t_x}{\partial y} \cdot dz = \frac{\partial^2 M_x}{\partial x^2} + 2 \cdot \frac{\partial^2 D}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2}.$$

Combining this relation with III, we find:

$$\begin{aligned} \frac{\partial^2 M_x}{\partial x^2} + 2 \cdot \frac{\partial^2 D}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + S_x h \cdot \frac{\partial^2 w}{\partial x^2} + 2 \cdot T_z h \cdot \frac{\partial^2 w}{\partial x \partial y} + S_y h \cdot \frac{\partial^2 w}{\partial y^2} \\ - X h \cdot \frac{\partial w}{\partial x} - Y h \cdot \frac{\partial w}{\partial y} - k w = 0. \end{aligned}$$

Using at last the relations:

$$s_x = - \frac{E \cdot m^2}{m^2 - 1} \cdot z \left( \frac{\partial^2 w}{\partial x^2} + \frac{1}{m} \cdot \frac{\partial^2 w}{\partial y^2} \right)$$

$$s_y = - \frac{E \cdot m^2}{m^2 - 1} \cdot z \left( \frac{\partial^2 w}{\partial y^2} + \frac{1}{m} \cdot \frac{\partial^2 w}{\partial x^2} \right)$$

$$t_z = - \frac{E \cdot m^2}{m^2 - 1} \cdot z \cdot \frac{\partial^2 w}{\partial x \partial y} \cdot \frac{m - 1}{m}$$

and putting  $\frac{E \cdot m^2}{m^2 - 1} \cdot \frac{h^3}{12} = EJ$ , the equation:

$$EJ \left\{ \frac{\partial^4 w}{\partial x^4} + 2 \cdot \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} \right\} - S_x h \cdot \frac{\partial^2 w}{\partial x^2} - 2 T_z h \cdot \frac{\partial^2 w}{\partial x \partial y} - S_y h \cdot \frac{\partial^2 w}{\partial y^2} + X h \cdot \frac{\partial w}{\partial x} + Y h \cdot \frac{\partial w}{\partial y} + k w = 0 \text{ (1).}$$

is obtained.

<sup>1)</sup> A specialisation of this equation can be found in "The math. theory of elasticity" by A. E. H. LOVE, 3rd edition 1920, p. 541.

12. *The girder, without volumetric-forces, elastically supported.*

When in the foregoing equation the differential-quotients with respect to  $y$ , as well as  $X$  and  $Y$  are put equal to zero, and if  $-S_x h$  be replaced by the total compressive force  $P$ , we fall back on the well-known equation

$$EJ \cdot \frac{d^4 w}{dx^4} + P \cdot \frac{d^2 w}{dx^2} + kw = 0$$

for the elastically supported girder.

13. *Vertical strut, subjected to its own weight.*

Again specialising and putting:  $Xh = -\frac{G}{1}$  ( $G$  = total weight of strut), and  $S_x h = -\frac{G(1-x)}{1}$  ( $x$  to be measured from the clamped end) we find:

$$EJ \cdot \frac{d^4 w}{dx^4} + \frac{G(1-x)}{1} \cdot \frac{d^2 w}{dx^2} - \frac{G}{1} \cdot \frac{dw}{dx} = 0 \text{ 1).}$$

14. *Buckling of a girder, clamped at one side.*

We examine a girder with a rectangular cross-section (height  $= h$ , breadth  $= b$ ,  $h \gg b$ ). The system of coordinates is placed in such a manner, that the  $x$ -axis coincides with the axis of the girder, and that the dimensions  $h$  and  $b$  of the cross-section are measured along the axis  $y$  and  $z$  respectively. The origin of the system of coordinates coincides with the centre of the clamped cross-section. The loading force  $P$ , acting in the centre of the free-end section, is parallel to the  $y$ -axis.

Then we have:

$$\left. \begin{aligned} S_x &= -\frac{12}{b \cdot h^3} \cdot Py(1-x) \\ T_z &= +\frac{6P}{b \cdot h^3} \cdot \left\{ \left(\frac{h}{2}\right)^2 - y^2 \right\} \\ T_x = T_y = S_y = S_z &= 0. \end{aligned} \right\} \quad \dots \quad (17)$$

The kinematic scheme of the girder will be described with the aid of two quantities  $w_0$  and  $\tau$ . The first of them gives the deflection of the axis of the beam in the  $z$ -direction; the second the angle of rotation of any cross-section about the  $x$ -axis. The displacements  $u, v, w$  of any point of the girder are then determined by the relations:

$$\left. \begin{aligned} u &= -\frac{\partial w_0}{\partial x} \cdot z - yz \cdot \frac{\partial \tau}{\partial x} \\ v &= -\tau \cdot z \\ w &= +w_0 + \tau \cdot y \end{aligned} \right\} \quad \dots \quad (17a)$$

<sup>1)</sup> Compare A. E. H. LOVE, p. 431-432.

as can easily be seen by using the kinematic scheme of the flat plate, and by remembering that the deflection of the axis of the girder in the direction  $y$  is negligible with respect to the other one, in the direction  $z$ . The quantities  $w_0$  and  $\tau$ , as a matter of fact, only depend on  $x$ .

The equations (3a), (3b), and (4) give:

$$\left. \begin{aligned} \omega_x &= \tau & f_x &= 0 & 2 \cdot r_x &= 0 \\ \omega_y &= -\frac{dw_0}{dx} - y \cdot \frac{dt}{dx}, \quad f_y &= 0 & 2 \cdot r_y &= 0 \\ \omega_z &= 0 & f_z &= -z \cdot \frac{dt}{dx}, \quad 2 \cdot r_z &= -S_x z \cdot \frac{dt}{dy} + \\ & & & & + T_z \cdot z \left( \frac{d^2 w_0}{dx^2} + y \cdot \frac{d^2 \tau}{dx^2} \right). \end{aligned} \right\} (17^b)$$

The variations  $\delta u$ ,  $\delta v$ ,  $\delta w$  are

$$\left. \begin{array}{l} \delta u = - \frac{\partial \delta w_0}{\partial x} \cdot z - y \cdot z \cdot \frac{\partial \delta \tau}{\partial x} \\ \delta v = - z \cdot \delta \tau \\ \delta w = + \delta w_0 + y \cdot \delta \tau. \end{array} \right\} \quad \dots \quad 17^c).$$

Hence, the equation (11), (after partial integration) can be written as follows:

$$\delta W = \int dx \left\{ \delta w_0 \iint \left( z \cdot \frac{\partial L}{\partial x} + N \right) dy \cdot dz + \delta \tau \iint \left( Ny + yz \cdot \frac{\partial L}{\partial x} - Mz \right) dy \cdot dz \right\} - \left[ \delta w_0 \iint Lz \, dy \, dz \right]_{x_1}^{x_2} - \left[ \delta \tau \iint Lyz \, dy \, dz \right]_{x_1}^{x_2} = 0.$$

Putting  $\delta r = 0$  everywhere, and  $\delta w_0 = 0$  at the ends of the girder we find:

$$\int \int \left( N + z \cdot \frac{\partial L}{\partial x} \right) dy dz = 0.$$

Putting  $\delta w_0 = 0$  everywhere, and  $\delta \tau = 0$  at the ends of the girder:

$$\iint \left( Ny + yz \cdot \frac{\partial L}{\partial x} - Mz \right) dy dz = 0.$$

Moreover the expressions between the [ ] brackets must vanish, so that:

$$\iint L \cdot z \, dy \, dz = \text{const.}$$

$$\iint L yz \, dy \, dz = \text{const.}$$

and therefore

$$\left. \begin{aligned} \iint \frac{\partial L}{\partial x} \cdot z \, dy \, dz &= 0 \\ \iint \frac{\partial L}{\partial x} \cdot yz \, dy \, dz &= 0 \end{aligned} \right\} \quad \dots \quad (17^d)$$

Consequently

$$\iint N \, dy \, dz = 0 \quad \dots \quad (I)$$

$$\iint (Ny - Mz) \, dy \, dz = 0 \quad \dots \quad (II)$$

15. Now we have to determine the functions  $L$ ,  $M$  and  $N$ , and find

$$L = \frac{\partial s_x}{\partial x} + \frac{\partial t_z}{\partial y} + \frac{\partial t_y}{\partial z} - \frac{\partial r_z}{\partial y}$$

$$M = \frac{\partial t_z}{\partial x} + \frac{\partial s_y}{\partial y} + \frac{\partial t_x}{\partial z} + \frac{\partial r_z}{\partial x}$$

$$N = \frac{\partial t_y}{\partial x} + \frac{\partial t_x}{\partial y} + \frac{\partial s_z}{\partial z} + 2 T_z \cdot \frac{dt}{dx} + S_x \left( \frac{d^2 w_0}{dx^2} + y \cdot \frac{d^2 \tau}{dx^2} \right),$$

so that equation I becomes:

$$\left. \begin{aligned} \iint \frac{\partial t_y}{\partial x} \cdot dy \, dz + \iint \frac{\partial t_x}{\partial y} \cdot dy \, dz + \iint \frac{\partial s_z}{\partial z} \cdot dy \, dz + 2 \iint T_z \cdot \frac{\partial \tau}{\partial x} \cdot dy \, dz \\ + \iint S_x \left( \frac{d^2 w_0}{dx^2} + y \cdot \frac{d^2 \tau}{dx^2} \right) = 0 \end{aligned} \right\} \quad (I)$$

Partial integration, leads, with due regard to the conditions:  $t_x = s_z = 0$  for  $y = \pm \frac{h}{2}$ , resp.  $z = \pm \frac{b}{2}$ , to

$$\iint \frac{\partial t_y}{\partial x} \cdot dy \, dz + 2 \cdot \frac{dt}{dx} \cdot P - \frac{d^2 \tau}{dx^2} \cdot (1-x) P = 0 \quad \dots \quad (I)$$

Equation II takes the form:

$$\left. \begin{aligned} \iint y \cdot \frac{\partial t_y}{\partial x} \cdot dy \, dz + \iint y \cdot \frac{\partial t_x}{\partial y} \cdot dy \, dz + \iint S_x \left( \frac{d^2 w_0}{dx^2} + y \cdot \frac{d^2 \tau}{dx^2} \right) y \, dy \, dz \\ - \iint \frac{\partial t_z}{\partial x} \cdot z \, dy \, dz - \iint \frac{\partial t_x}{\partial z} \cdot z \, dy \, dz - \iint \frac{\partial r_z}{\partial y} \cdot z \, dy \, dz = 0. \end{aligned} \right\} \quad (II)$$

In order to determine the integrals

$$\iint y \cdot \frac{\partial t_y}{\partial x} \cdot z \, dy \, dz \quad \text{and} \quad \iint \frac{\partial t_y}{\partial x} \cdot dy \, dz$$

we use equation (17<sup>d</sup>).

In consequence of the fact, that

$$\frac{\partial L}{\partial x} = \frac{\partial^2 s_x}{\partial x^2} + \frac{\partial^2 t_z}{\partial x \partial y} + \frac{\partial^2 t_y}{\partial x \partial z} - \frac{\partial^2 r_z}{\partial x \partial y}$$

we find by transforming in the usual manner  $\iint \frac{\partial L}{\partial x} \cdot z \, dy \, dz = 0$ :

$$\iint \frac{\partial t_y}{\partial x} \cdot dy \, dz = \iint z \cdot \frac{\partial^2 s_x}{\partial x^2} \cdot dy \, dz - \iint z \cdot \frac{\partial^2 r_z}{\partial x \partial y} \cdot dy \, dz \quad \dots \quad (\text{III})$$

From  $\iint \frac{\partial L}{\partial x} \cdot yz \, dy \, dz = 0$  we derive:

$$\begin{aligned} \iint y \cdot \frac{\partial t_y}{\partial x} \cdot dy \, dz + \iint z \cdot \frac{\partial t_z}{\partial x} \cdot dy \, dz = & \iint yz \cdot \frac{\partial^2 s_x}{\partial x^2} \cdot dy \, dz - \\ & - \iint yz \cdot \frac{\partial^2 r_z}{\partial x \partial y} \cdot dy \, dz \end{aligned} \quad \left. \right\} \quad (\text{IV})$$

At the other hand the torsional moment of the cross-section is equal to:

$$M_w = \iint t_z \cdot z \cdot dy \, dz - \iint t_y \cdot y \cdot dy \, dz,$$

so that

$$\iint \frac{\partial t_z}{\partial x} \cdot z \cdot dy \, dz - \iint \frac{\partial t_y}{\partial x} \cdot y \cdot dy \, dz = \frac{dM_w}{dx} \quad \dots \quad (\text{V})$$

Substituting (III) in (I), we find:

$$\iint z \cdot \frac{\partial^2 s_x}{\partial x^2} \cdot dy \, dz - \iint z \cdot \frac{\partial^2 r_z}{\partial x \partial y} \cdot dy \, dz + 2 \cdot \frac{d\tau}{dx} \cdot P - P(1-x) \cdot \frac{d^2 \tau}{dx^2} = 0 \quad \dots \quad (\text{I}')$$

Combining (IV) and (II), we have:

$$\begin{aligned} \iint yz \cdot \frac{\partial^2 s_x}{\partial x^2} \cdot dy \, dz - 2 \cdot \iint \frac{\partial t_z}{\partial x} \cdot z \cdot dy \, dz - \iint yz \cdot \frac{\partial^2 r_z}{\partial x \partial y} \cdot dy \, dz - \\ - \iint \frac{\partial r_z}{\partial x} \cdot z \cdot dy \, dz + \iint yS_x \cdot \frac{d^2 w_0}{dx^2} \cdot dy \, dz = 0. \end{aligned} \quad \left. \right\} \quad (\text{II}')$$

From (V) and (IV) it follows that:

$$2 \cdot \iint z \cdot \frac{\partial t_z}{\partial x} \cdot dy \, dz = \iint yz \left( \frac{\partial^2 s_x}{\partial x^2} - \frac{\partial^2 r_z}{\partial x \partial y} \right) dy \, dz + \frac{dM_w}{dx}.$$

If now  $r_z$  is replaced by the value, determined by equation (17<sup>b</sup>), the equations (I) and (II) become:

$$\iint z \cdot \frac{\partial^2 s_x}{\partial x^2} \cdot dy \, dz + 2 \cdot P \cdot \frac{d\tau}{dx} - P(1-x) \cdot \frac{d^2 \tau}{dx^2} = 0 \quad \dots \quad (\text{I}'')$$

$$+ \frac{dM_w}{dx} + \iint \frac{\partial r_z}{\partial x} \cdot z \cdot dy \, dz + P(1-x) \cdot \frac{d^2 w_0}{dx^2} = 0 \quad \dots \quad (\text{II}'')$$

According to equation (17b), we have

$$2 \cdot z \cdot \frac{\partial r_z}{\partial x} = T_z \cdot z^2 \cdot \frac{d^3 w_0}{dx^3}$$

or on account of (17)

$$2 \cdot \iint z \cdot \frac{\partial r_z}{\partial x} \cdot dy \, dz = \frac{d^3 w_0}{dx^3} \cdot \frac{6 \cdot P}{bh^3} \cdot \iint \left( \frac{h^2}{4} - y^2 \right) \cdot z^2 \cdot dy \, dz$$

resp.:

$$\iint z \cdot \frac{\partial r_z}{\partial x} \cdot dy \, dz = \frac{Pb^2}{24} \cdot \frac{d^3 w_0}{dx^3}$$

so that (II) can be reduced to

$$\frac{dM_w}{dx} + P(1-x) \cdot \frac{d^2 w_0}{dx^2} + \frac{Pb^2}{24} \cdot \frac{d^3 w_0}{dx^3} = 0.$$

Finally we have to make use of the relations

$$s_x = +E \cdot \frac{\partial u}{\partial x} = -Ez \left( \frac{d^2 w_0}{dx^2} + y \cdot \frac{d^2 \tau}{dx^2} \right)$$

and

$$M_w = G \cdot C \cdot \frac{d\tau}{dx}$$

(where  $C$  is a constant, depending on the dimensions of the cross-section).

Doing so, and putting  $\frac{b \cdot h^3}{12} = J$ , we find the simultaneous differential equations:

$$EJ \cdot \frac{d^4 w_0}{dx^4} + P(1-x) \cdot \frac{d^2 \tau}{dx^2} - 2 \cdot P \cdot \frac{d\tau}{dx} = 0$$

$$GC \cdot \frac{d^2 \tau}{dx^2} + P(1-x) \cdot \frac{d^2 w_0}{dx^2} + \frac{Pb^2}{24} \cdot \frac{d^3 w_0}{dx^3} = 0.$$

which, with the exception of the term  $\frac{Pb^2}{24} \cdot \frac{d^3 w_0}{dx^3}$ , agree with those found by other investigators. Along other lines TIMOSCHENKO<sup>1)</sup> introduced a similar term, when he made experiments with I-girders and stated a discrepancy between the usual theory and his results.

<sup>1)</sup> TIMOSCHENKO, Sur la stabilité des systèmes élastiques. Paris 1913, or Annales des ponts et chaussées, fasc. 3, 4 and 5 1913.

## 16. The two-dimensional problem in polar coordinates.

If the state of stress of the body under consideration is independent of  $z$  (so that  $\omega_x = \omega_y = 0$ ,  $f_x = f_y = 0$ ), and if the volumetric forces are negligible, the problem of elastic stability is governed by the equations:

$$L \equiv \frac{\partial s_x}{\partial x} + \frac{\partial t_z}{\partial y} - T_z \cdot \frac{\partial \omega_z}{\partial x} - S_y \cdot \frac{\partial \omega_z}{\partial y} - \frac{\partial r_z}{\partial y} = 0 \quad \dots \quad (I)$$

$$M \equiv \frac{\partial t_z}{\partial x} + \frac{\partial s_y}{\partial y} + S_x \cdot \frac{\partial \omega_z}{\partial x} + T_z \cdot \frac{\partial \omega_z}{\partial y} + \frac{\partial r_z}{\partial x} = 0 \quad \dots \quad (II)$$

$$2 \cdot r_z = (S_x - S_y) f_z + T_z \left( \frac{\partial v}{\partial y} - \frac{\partial u}{\partial x} \right). \quad \dots \quad (III)$$

Though, as a matter of fact, it would be possible to transform these equations in polar coordinates in a formal manner, we shall give here the direct deduction of the main equations. The polar coordinates of a point being  $(\varrho, \varphi)$ , the radial and tangential displacements  $u$  and  $v$ , we have:

$$f_z = \frac{1}{2} \cdot \left( \frac{\partial v}{\partial \varrho} - \frac{v}{\varrho} + \frac{1}{\varrho} \cdot \frac{\partial u}{\partial \varphi} \right)$$

$$\omega_z = \frac{1}{2\varrho} \cdot \left\{ \frac{\partial (v\varrho)}{\partial \varrho} - \frac{\partial u}{\partial \varphi} \right\}$$

$$2 \cdot r_z = (S_\varrho - S_\varphi) f_z + T_z \left( \frac{1}{\varrho} \cdot \frac{\partial v}{\partial \varphi} + \frac{u}{\varrho} - \frac{\partial u}{\partial \varrho} \right).$$

The relations (5) have to be replaced by

$$X_\varrho = S_\varrho + s_\varrho - T_z \omega_z$$

$$X_\varphi = T_z + t_z - r_z - S_\varphi \omega_z$$

$$Y_\varrho = T_z + t_z + r_z + S_\varrho \omega_z$$

$$Y_\varphi = S_\varphi + s_\varphi + T_z \cdot \omega_z$$

where obviously  $X_\varrho, X_\varphi, \dots$  signify stress-components in the directions  $\varrho$  and  $\varphi$ , related to surface-elements in state I, but acting on the corresponding surface-elements in state II.

The equations of equilibrium of an element in state II, originally lying between the two angles  $\varphi$  and  $\varphi + d\varphi$ , and the two circles  $(\varrho)$  and  $(\varrho + d\varrho)$ , take the form

$$\frac{1}{\varrho} \cdot \frac{\partial (X_\varrho \varrho)}{\partial \varrho} + \frac{1}{\varrho} \cdot \frac{\partial X_\varphi}{\partial \varphi} - \frac{Y_\varphi}{\varrho} = 0$$

$$\frac{1}{\varrho} \cdot \frac{\partial (Y_\varrho \varrho)}{\partial \varrho} + \frac{1}{\varrho} \cdot \frac{\partial Y_\varphi}{\partial \varphi} + \frac{X_\varphi}{\varrho} = 0$$

or

$$\frac{\partial X_p}{\partial \varrho} + \frac{1}{\varrho} \cdot \frac{\partial X_\varphi}{\partial \varphi} + \frac{1}{\varrho} \cdot (X_p - Y_\varphi) = 0$$

$$\frac{\partial Y_p}{\partial \varrho} + \frac{1}{\varrho} \cdot \frac{\partial Y_\varphi}{\partial \varphi} + \frac{1}{\varrho} \cdot (X_\varphi + Y_p) = 0.$$

Substituting in these equations the values of  $X_p$ ,  $X_\varphi$ ,  $Y_p$ ,  $Y_\varphi$  we obtain, with regard to the equations of equilibrium, holding for the element in state I:

$$\frac{\partial S_p}{\partial \varrho} + \frac{1}{\varrho} \cdot \frac{\partial T_z}{\partial \varphi} + \frac{1}{\varrho} \cdot (S_p - S_\varphi) = 0$$

$$\frac{\partial T_z}{\partial \varrho} + \frac{1}{\varrho} \cdot \frac{\partial S_\varphi}{\partial \varphi} + \frac{2 \cdot T_z}{\varrho} = 0,$$

as our final equations:

$$\frac{\partial s_p}{\partial \varrho} + \frac{1}{\varrho} \cdot \frac{\partial t_z}{\partial \varphi} + \frac{1}{\varrho} \cdot (s_p - s_\varphi) - T_z \cdot \frac{\partial \omega_z}{\partial \varrho} - \frac{S_\varphi}{\varrho} \cdot \frac{\partial \omega_z}{\partial \varphi} - \frac{1}{\varrho} \cdot \frac{\partial r_z}{\partial \varphi} = 0$$

$$\frac{\partial t_z}{\partial \varrho} + \frac{1}{\varrho} \cdot \frac{\partial s_\varphi}{\partial \varphi} + \frac{2 t_z}{\varrho} + S_p \cdot \frac{\partial \omega_z}{\partial \varrho} + \frac{T_z}{\varrho} \cdot \frac{\partial \omega_z}{\partial \varphi} + \frac{\partial r_z}{\partial \varrho} = 0.$$

They reduce, with  $T_z = 0$ , to

$$L \equiv \frac{\partial s_p}{\partial \varrho} + \frac{1}{\varrho} \cdot \frac{\partial t_z}{\partial \varphi} + \frac{1}{\varrho} \cdot (s_p - s_\varphi) - \frac{S_\varphi}{\varrho} \frac{\partial \omega_z}{\partial \varphi} - \frac{1}{\varrho} \cdot \frac{\partial r_z}{\partial \varphi} = 0. \quad \text{. . (I)}$$

$$M \equiv \frac{\partial t_z}{\partial \varrho} + \frac{1}{\varrho} \cdot \frac{\partial s_\varphi}{\partial \varphi} + \frac{2 t_z}{\varrho} + S_p \frac{\partial \omega_z}{\partial \varrho} + \frac{\partial r_z}{\partial \varrho} = 0. \quad \text{. . . . . (II)}$$

### 17. The circular ring under external pressure.

The kinematic scheme be characterised by

$$u = u_0$$

$$v = v_0 \frac{\varrho}{a} - \frac{1}{a} \cdot \frac{\partial u_0}{\partial \varphi} \cdot (\varrho - a);$$

$a$  is the average ring-radius,  $u, v$  are the displacements of an arbitrary point  $(\varrho, \varphi)$ ,  $u_0, v_0$  the displacements of the corresponding point  $(a, \varphi)$ .

We find consecutively:

$$\delta u = \delta u_0$$

$$\delta v = \frac{\varrho}{a} \delta v_0 - \frac{1}{a} \cdot \frac{\partial \delta u_0}{\partial \varphi} \cdot (\varrho - a)$$

$$f_z = 0 \quad r_z = 0$$

$$\frac{\partial \omega_z}{\partial \varphi} = \frac{1}{a} \cdot \left( \frac{\partial v_0}{\partial \varphi} - \frac{\partial^2 u_0}{\partial \varphi^2} \right)$$

and therefore:

$$L \equiv \frac{\partial s_\rho}{\partial \varrho} + \frac{1}{\varrho} \cdot \frac{\partial t_z}{\partial \varphi} + \frac{1}{\varrho} \cdot (s_\rho - s_\tau) - \frac{S_\varphi}{\varrho} \cdot \frac{\partial \omega_z}{\partial \varphi}$$

$$M \equiv \frac{\partial t_z}{\partial \varrho} + \frac{1}{\varrho} \cdot \frac{\partial s_\rho}{\partial \varphi} + \frac{2t_z}{\varrho}.$$

The equation

$$\iint (L \delta u + M \delta v) \varrho \, d\varphi \, d\varrho = 0,$$

which now governs the problem, therefore can be written as follows:

$$\int d\varphi \left[ \delta u_0 \int L \varrho \, d\varrho + \delta v_0 \int \frac{M \varrho^2}{a} \cdot d\varrho - \frac{\partial \delta u_0}{a \partial \varphi} \cdot \int M \varrho (\varrho - a) \, d\varrho \right] = 0$$

$$\int d\varphi \left[ \delta u_0 \left\{ \int L \varrho \, d\varrho + \int \frac{\partial M}{a \partial \varphi} \varrho (\varrho - a) \, d\varrho \right\} + \delta v_0 \int \frac{M \varrho^2}{a} \, d\varrho \right] = 0.$$

Consequently

$$\int L \varrho \, d\varrho + \int \frac{\partial M}{\partial \varphi} \cdot \frac{(\varrho - a) \varrho}{a} \, d\varrho = 0 \quad \dots \dots \dots \quad (I)$$

$$\int M \varrho^2 \, d\varrho = 0. \quad \dots \dots \dots \quad (II)$$

The latter of these equations becomes

$$\int \frac{\partial t_z}{\partial \varrho} \varrho^2 \, d\varrho + \int \frac{\partial s_\rho}{\partial \varphi} \varrho \, d\varrho + 2 \int t_z \varrho \, d\varrho = 0.$$

Now we have

$$s_\tau = E \cdot \left\{ \frac{u_0}{\varrho} + \frac{1}{a} \cdot \frac{\partial v_0}{\partial \varphi} - \frac{\partial^2 u_0}{\partial \varphi^2} \cdot \frac{\varrho - a}{a \varrho} \right\}$$

$$\frac{\partial s_\rho}{\partial \varphi} = E \cdot \left\{ \frac{1}{\varrho} \cdot \frac{\partial u_0}{\partial \varphi} + \frac{1}{a} \cdot \frac{\partial^2 v_0}{\partial \varphi^2} - \frac{\partial^3 u_0}{\partial \varphi^3} \cdot \frac{\varrho - a}{a \varrho} \right\}$$

$$\int \varrho \cdot \frac{\partial s_\rho}{\partial \varphi} \, d\varrho = Eh \cdot \frac{\partial u_0}{\partial \varphi} + Eh \cdot \frac{\partial^2 v_0}{\partial \varphi^2}$$

$$\int \varrho^2 \cdot \frac{\partial t_z}{\partial \varrho} \, d\varrho = (t_z \varrho^2)^{\frac{a}{a-\frac{h}{2}} + \frac{h}{2}} - 2 \int t_z \varrho \, d\varrho = -2 \int t_z \varrho \, d\varrho$$

as  $t_z = 0$  for  $\varrho = a + \frac{h}{2}$  and  $a - \frac{h}{2}$ .

Therefore the equation reduces to

$$\frac{\partial u_0}{\partial \varphi} + \frac{\partial^2 v_0}{\partial \varphi^2} = 0 \quad \text{or} \quad u_0 = -\frac{\partial v_0}{\partial \varphi}.$$

It follows that the centre-line of the ring does not change its length. Equation (I) passes into:

$$\int \varrho \frac{\partial s_\rho}{\partial \varrho} \cdot d\varrho + \int \frac{\partial t_z}{\partial \varphi} \cdot d\varrho + \int s_\rho \cdot d\varrho - \int s_\varphi d\varrho - \frac{\partial \omega_z}{\partial \varphi} \cdot \int S_\varphi d\varrho \\ + \int \frac{\partial^2 t_z}{\partial \varphi \partial \varrho} \varrho \cdot \frac{\varrho - a}{a} \cdot d\varrho + \int \frac{\partial^2 s_\rho}{\partial \varphi^2} \cdot \frac{\varrho - a}{a} \cdot d\varrho + 2 \int \frac{\partial t_z}{\partial \varphi} \cdot \frac{\varrho - a}{a} d\varrho = 0$$

where

$$\int S_\varphi d\varrho = \left( a + \frac{h}{2} \right) \cdot P_\rho .$$

The boundary conditions are

$$\varrho = a + \frac{h}{2} \begin{cases} P_\varphi = 0 \\ s_\rho = P_\rho \left( \frac{1}{\varrho} \cdot \frac{\partial v}{\partial \varphi} + \frac{u}{\varrho} \right) \end{cases}$$

which may be written as

$$\varrho = a + \frac{h}{2} \begin{cases} P_\varphi = 0 \\ s_\rho = -P_\rho \cdot \frac{h}{2a(a + h/2)} \cdot \left( u_0 + \frac{\partial^2 u_0}{\partial \varphi^2} \right) \end{cases} .$$

in consequence of the fact, that

$$\frac{1}{\varrho} \frac{\partial v}{\partial \varphi} + \frac{u}{\varrho} = \frac{1}{\varrho} \left\{ u_0 + \frac{\varrho}{a} \cdot \frac{\partial v_0}{\partial \varphi} - \frac{1}{a} \cdot \frac{\partial^2 u_0}{\partial \varphi^2} \cdot (\varrho - a) \right\} \\ = \frac{u_0}{a + \frac{h}{2}} + \frac{1}{a} \cdot \frac{\partial v_0}{\partial \varphi} - \frac{1}{a} \cdot \frac{\partial^2 u_0}{\partial \varphi^2} \cdot \frac{h}{2(a + \frac{h}{2})} \\ = -\frac{h}{2a(a + h/2)} \cdot \left( u_0 + \frac{\partial^2 u_0}{\partial \varphi^2} \right) .$$

Transforming equation (I), we find successively:

$$\int \varrho \frac{\partial s_\rho}{\partial \varrho} \cdot d\varrho + \int s_\rho \cdot d\varrho = (\varrho \cdot s_\rho)_{a-h/2}^{a+h/2} \\ = (a + h/2) \cdot s_{a+h/2} = -P_\rho \cdot \frac{h}{2a} \cdot \left( u_0 + \frac{\partial^2 u_0}{\partial \varphi^2} \right)$$

$$\int s_\varphi d\varrho = Eu_0 \int \frac{d\varrho}{\varrho} + \frac{E}{a} \frac{\partial v_0}{\partial \varphi} \int d\varrho - E/a \frac{\partial^2 u_0}{\partial \varphi^2} \int \frac{\varrho - a}{\varrho} d\varrho$$

and putting

$$\beta = 1n \frac{a+h/2}{a-h/2} \quad , \quad \frac{\partial v_0}{\partial \varphi} = -u_0$$

$$\int s_r d\varphi = Eu_0(\beta - h/a) + E \frac{\partial^2 u_0}{\partial \varphi^2}(\beta - h/a) = E(\beta - h/a) \left( u_0 + \frac{\partial^2 u_0}{\partial \varphi^2} \right)$$

$$\frac{\partial \omega_z}{\partial \varphi} \int S_z d\varphi = P_z \left( 1 + \frac{h}{2a} \right) \left( \frac{\partial v_0}{\partial \varphi} - \frac{\partial^2 u_0}{\partial \varphi^2} \right) = -P_z \left( 1 + \frac{h}{2a} \right) \left( u_0 + \frac{\partial^2 u_0}{\partial \varphi^2} \right)$$

$$\int \frac{\varrho^2 - \varrho a}{a} \cdot \frac{\partial^2 t_z}{\partial \varrho \partial \varphi} \cdot d\varphi = \left\{ \frac{\partial t_z}{\partial \varphi} \cdot \frac{\varrho^2 - a\varrho}{a} \right\}_{a-h/2}^{a+h/2} - \int \frac{\partial t_z}{\partial \varphi} \cdot \frac{2\varrho - a}{a} \cdot d\varphi$$

$$= -2 \int \frac{\partial t_z}{\partial \varphi} \cdot \frac{\varrho - a}{a} - \int \frac{\partial t_z}{\partial \varphi} \cdot d\varphi$$

$$\int \frac{\partial^2 s_r}{\partial \varphi^2} \cdot \frac{\varrho - a}{a} \cdot d\varphi = -E \cdot \frac{\partial^2 u_0}{\partial \varphi^2} \int \frac{(\varrho - a)^2}{a^2 \varrho} d\varphi - E \cdot \frac{\partial^4 u_0}{\partial \varphi^4} \int \frac{(\varrho - a)^2}{a^2 \varrho} d\varphi$$

$$= -E(\beta - h/a) \left( \frac{\partial^2 u_0}{\partial \varphi^2} + \frac{\partial^4 u_0}{\partial \varphi^4} \right).$$

The final result therefore is:

$$-P_z \cdot \frac{h}{2a} \left( u_0 + \frac{\partial^2 u_0}{\partial \varphi^2} \right) - E(\beta - h/a) \left( u_0 + \frac{\partial^2 u_0}{\partial \varphi^2} \right) + P_z \left( 1 + \frac{h}{2a} \right) \left( u_0 + \frac{\partial^2 u_0}{\partial \varphi^2} \right)$$

$$- E(\beta - h/a) \left\{ \frac{\partial^2 u_0}{\partial \varphi^2} + \frac{\partial^4 u_0}{\partial \varphi^4} \right\} = 0$$

or

$$\frac{\partial^4 u_0}{\partial \varphi^4} + 2 \cdot \frac{\partial^2 u_0}{\partial \varphi^2} - \frac{P_z}{E(\beta - a/h)} \cdot \left( u_0 + \frac{\partial^2 u_0}{\partial \varphi^2} \right) + u_0 = 0$$

or

$$\frac{\partial^4 u_0}{\partial \varphi^4} + \frac{\partial^2 u_0}{\partial \varphi^2} (2 + \gamma) + u_0 (1 + \gamma) = 0,$$

with

$$\gamma = -\frac{P_z}{E(\beta - h/a)} = -\frac{P_z \cdot a^3}{EJ}.$$

The solutions  $\cos \varphi$  and  $\sin \varphi$  of this equation are non-essential, as they relate to displacements of the ring as a whole. The remaining solutions  $\cos \sqrt{1+\gamma} \varphi$ ,  $\sin \sqrt{1+\gamma} \varphi$  have to be periodical with the period  $2\pi$ . The smallest value for  $\gamma$  therefore follows from

$$\sqrt{1+\gamma} = 2 \quad , \quad \gamma = 3.$$

which is in accordance with results already known.

**Physics.** — *The spectra of ionized Neon ( $Ne II$ ) and ionized Argon ( $Ar II$ ).* (Second Communication.) By T. L. DE BRUIN. (Communicated by Prof. P. ZEEMAN.)

(Communicated at the meeting of March 31, 1928).

### 1. *Introduction.*

In a former paper in these Proceedings<sup>1)</sup> I have classified in a term-scheme a number of 180 lines of the spectrum of ionized Neon ( $Ne II$ ). The spectralstructure has been found in agreement with the theory of complex spectra developed by HEISENBERG, RUSSEL, SAUNDERS, PAULI and especially by HUND. It has been pointed out that the structure is analogous with the spectralstructure of a neutral halogenatom (Fluorine) as could be expected according to the theory of BOHR and the displacement-law of KOSSEL and SOMMERFELD. The wavelengthmeasurements used for that analysis were taken from L. BLOCH, E. BLOCH and G. DÉJARDIN who produced the spectrum with the electrodeless discharge.

Now I have made some new experiments on the spectra of the ionized inert gases  $Ne II$  and  $Ar II$ . The spectra have been produced by passing heavy condensed discharges through a narrow vacuumcapilar ( $\frac{3}{4}$  mm). For more experimental details see a note in the Zeitschr. f. Physik.<sup>2)</sup>. A comparison between the deep quartetterms of  $Ne II$  and  $Ar II$  forms the substance of the present paper.

### 2. *The spectrum of ionized Argon.*

In the same year (1895) that RAYLEIGH and RAMSAY had detected the element argon, CROOKES investigated the spectrum of it. CROOKES found that two spectra could be produced, the "red" and the "blue" spectrum. The blue spectrum was very intensive in the case of condensed discharges. Wave-length measurements of this blue spectrum have been made by

<sup>1)</sup> T. L. DE BRUIN: These Proceedings **31**, 2, 1928.

<sup>2)</sup> T. L. DE BRUIN: Zeitschr. f. Physik. **44**, 157. 1927. **46**, 856. 1928.

KAYSER<sup>1)</sup> and EDER and VALENTA<sup>2)</sup>. Especially the measurements of KAYSER to  $\lambda$  3000 given in 0.001 Å are very useful for the analysis. The data of EDER and VALENTA are not so exact but extend to  $\lambda$  2000. L. BLOCH, E. BLOCH and G. DÉJARDIN<sup>3)</sup> have investigated which lines are due to the higher states of ionization. Little has been known on the structure of the blue Argonspectrum<sup>4)</sup>. Only PAULSON<sup>5)</sup> found that different lines could be ordered in groups with the differences :

$$\Delta\nu: 844.49; 1611.33; 149.55 \text{ and } 153.97.$$

The analysis of the  $Ne\ II$  spectrum was the startingpoint for the analysis of the  $Ar\ II$  spectrum. The key for the analysis of  $Ne\ II$  was the identification of the following group of lines :

$Ne\ II$	$4P_1$	299.1	$4P_2$	517.7	$4P_3$
$4S_2$	4.3028.85 33006.2		6.3001.646 33305.3		7.2955.735 33822.7

This group is in the case of ionized Argon :

$Ar\ II$	$4P_1$	515.70	$4P_2$	844.40	$4P_3$
$4S_2$	7.3928.599 25447.24		8.3850.565 25962.88		9.3729.300 26807.09

In this note we shall compare the quartetterms in both spectra, whereas in a following paper we shall give an account of the doublet terms, the intercombinations, the absolute termvalues, etc.

### 3. Theoretical scheme of terms.

Theoretically, according to the theory of complex spectra the following terms may be deduced from the atomic configuration :

<sup>1)</sup> H. KAYSER: *Astrophys. Journ.* **4**, 1. 1896. *Sitzber. Akad. Berlin.* 1896. Nr. 24; *Handb. d. Spektr.* V. pag. 61. These measurements have been used.

<sup>2)</sup> EDER et VALENTA: *Denkschr. Wien. Akad.* **64**, 1. 1897. *Beitr. z. Photochem.* 1904.

<sup>3)</sup> L. BLOCH, E. BLOCH et G. DÉJARDIN: *Ann. d. Phys.* **2**, 461. 1924.

<sup>4)</sup> K. W. MEISSNER investigated the structure of the "red" spectrum due to the neutral atom.

<sup>5)</sup> PAULSON: *Astrophys. Journ.* **41**, 75. 1915.

TABLE I. Theoretical termschemes Neon II and Argon II (Deep terms).

nk	Atomic configuration							Symbol	Basicterm: $^3P^1$ )		
	1s 2s 2p			3s 3p 3d		4s 4p 4d 4f			Terms.		
	Quartet	Doublet									
Neon II	2	2	5					$s^2 p^5$		P	
	2	2	4	1				$s^2 p^4 . 3s$	P	P	
	2	2	4		1			$s^2 p^4 . 3p$	D P S	D P S	
	2	2	4			1		$s^2 p^4 . 3d$	F D P	F D P	
	2	2	4				1	$s^2 p^4 . 4s$	P	P	
Argon II	2	2	6	2	5			$s^2 p^5$		P	
	2	2	6	2	4	1		$s^2 p^4 . 3d$	F D P	F D P	
	2	2	6	2	4		1	$s^2 p^4 . 4s$	P	P	
	2	2	6	2	4			$s^2 p^4 . 4p$	D P S	D P S	
	2	2	6	2	4			$s^2 p^4 . 4d$	F D P	F D P	

The termarrangement of  $Ar\ II$  is similar to that of  $Ne\ II$  already reported by us, except for one group of terms. In  $Ne\ II$  only terms involving a displaced  $s$  electron exist in the group of terms above the fundamental  $2P_{21}$  state. In  $Ar\ II$  however, an excited  $3d$  electron gives a group of terms 2, 4 ( $F, D, P$ ).

#### 4. Analogous multiplets. Termtable, Identification of terms.

The following table 2 gives the analogous multiplets arising from the combinations between the deep quartetterms of  $Ne\ II$  and  $Ar\ II$ . In both spectra <sup>2)</sup> in the  $^4P_{321}$  term the interval rule holds very well but in the  $^4P'_{321}$ -term the ratio is anomalous. In the  $^4P_{321}$ -term the three components go to the components of the basicterm  $^3P_{012}$  but in the  $^4P'_{321}$ -term the components  $^4P_1$  and  $^4P_2$  go to  $^3P_1$  while  $^4P_3$  goes to  $^3P_2$ . In both spectra the  $^4D_{12}$  and  $^4F_{23}$  (in  $Ne\ II$  144.1 and 145.5 in  $Ar\ II$  260.5 and 266.3) have nearly the same value. These termseparations go to the component  $^3P_0$  of the basicterm. Table 4 shows that in  $Ar\ II$  terms are present of the configuration  $(s^2 p^4) 3d$ . A numerical comparison between wavenumbers of the analogous multiplets shows that the ratio diminishes for higher terms (Table 5). Table 3 gives some multiplets in both spectra which identification is not quite sure. Table 6 gives the relative term values in  $Ne\ II$  and  $Ar\ II$ . The last column gives all the theoretical terms.

<sup>1)</sup> It is possible that also terms from the metastable states  $^1D$  and  $^1S$  are present.

<sup>2)</sup> This is also in the  $FI$  spectrum. See T. L. DE BRUIN. These Proceedings **30**, p. 20 and 944, 1927; Zs. f. Phys. **38**, 94, 1926.

TABLE 2. Neon II.  $3s\ 4P_{123} - 3p\ 4P_{123}$ ;  $3s\ 4P_{123} - 3p\ 4D_{1234}$ ;  $3s\ 4P_{123} - 3p\ 4S_2$ .

	$3s$ $3p$	$4P_1$	299.1	$4P_2$	517.7	$4P_3$
1.	$4P'_1$	5. 3751.26 26650.2	299.0	7. 3709.643 26949.1	—	—
	182.6	8. 3777.162 26467.4	299.2	7. 3734.94 26766.6	517.6	9. 3664.089 27284.2
	222.6	$4P'_3$ —		8. 3766.286 26543.8	517.8	10. 3694.22 27061.6
2.	$4D_1$	5. 3344.43 29891.9	299.1	3. 3311.30 30191.0	—	—
	144.1	$4D_2$ 5. 3360.63 29747.8	299.2	5. 3327.162 30047.0	517.9	2. 3270.79 30564.9
	249.6	$4D_3$ —		7. 3355.05 29797.4	517.7	7. 3297.74 30315.1
3.	$4D_4$	—		—	—	10. 3334.837 29977.8
	$4S_2$	4. 3028.85 33006.2	299.1	6. 3001.646 33305.3	517.4	7. 2955.735 33822.7

Argon II.  $4s\ 4P_{123} - 4p\ 4P_{123}$ ;  $4s\ 4P_{123} - 4p\ 4D_{1234}$ ;  $4s\ 4P_{123} - 4p\ 4S_2$ .

	$4s$ $4p$	$4P_1$	515.70	$4P_2$	844.40	$4P_3$
I.	$4P'_1$	4. 4972.16 <sup>1)</sup> 20106.4	515.85	8. 4847.783 20622.25	—	—
	357.30	$4P'_2$ 8. 5062.019 19749.47	515.60	6. 4933.226 20265.07	844.42	6. 4735.885 21109.49
	307.78	$4P'_3$ —		8. 5009.246 19957.53	844.03	10. 4805.993 20801.56
II.	$4D_1$	6. 4379.657 22826.44	515.72	3. 4282.894 23342.16	—	—
	260.31	$4D_2$ 4. 4430.185 22566.11	515.74	8. 4331.194 23081.85	844.36	3. 4178.344 23926.21
	494.40	$4D_3$ —		8. 4425.995 22587.23	844.38	8. 4266.524 23431.71
III.	$4D_4$	—	—	—	—	10. 4348.062 22992.30
	$4S_2$	7. 3928.599 25447.20	515.68	8. 3850.565 25962.88	844.21	9. 3729.300 26807.09

<sup>1)</sup> Measurement of the author.

TABLE 2 (Continued). Neon II.  $3d\ 4D_{1234} - 3p\ 4P_{123}$ ;  $3d\ 4D_{1234} - 3p\ D_{1234}$ .

	$3d\backslash 3p$	$4D'_1$	98.3	$4D'_2$	106.3	$4D'_3$	81.1	$4D'_4$
4.	$4P'_1$	3. 3045.58 32824.9	98.8	5. 3054.69 32727.1		—		—
	182.6							
	$4P'_2$	4. 3028.85 <sup>1)</sup> 33006.2	97.4	4. 3037.73 32909.8	106.3	6. 3047.57 32803.5		—
5.	222.5							
	$4P'_3$			3. 3017.34 33132.2	106.2	4. 3027.04 33026.0	81.0	5. 3034.47 <sup>5</sup> 32945.0
		—						

Argon II.  $4d\ 4D_{1234} - 4p\ 4P_{123}$ ;  $4d\ 4D_{1234} - 4p\ 4D_{1234}$ .

	$4d\backslash 4p$	$4D'_1$	107.50	$4D'_2$	188.43	$4D'_3$	121.85	$4D'_4$
IV.	$4P'_1$	5. 3521.950 28385.27	107.70	6. 3535.364 28277.57		—		—
	357.30							
	$4P'_2$	6. 3478.260 28741.81	107.27	5. 3491.290 28634.54	188.50	8. 3514.426 28446.04		—
V.	307.76							
	$4P'_3$			5. 3454.148 28954.44	188.37	8. 3476.776 28754.07	121.85	10. 3491.573 28632.22
		—						
VI.	$4D'_1$	( 3895.19 <sup>4)</sup> (25665.45)	107.49	1. 3911.571 25557.96		—		—
	260.31							
	$4D'_2$	1. 3856.060 25925.88	107.90	2. 3872.176 25817.98	188.22	2. 3900.613 25629.76		—
	494.40							
VII.	$4D'_3$			1. 3799.446 26312.18	188.25	3. 3826.826 26123.93	121.93	1. 3844.771 26002.00
	439.47							
	$4D'_4$					3. 3763.565 26563.03	121.56	6. 3780.868 26441.47

1) Coincidence with:  $4P_1 - 4S_2$ .2) Coincidence with:  $4S_2 - g\ 4P_1$ .

3) Probably coincidence, Intensity too much.

4) Wavelengths between brackets are calculated. Note added to proof: This line has been found on our plates. Observed: 1. 3895.22; 25665.2.

TABLE 2 (Continued). Neon II.  $3d\ 4F_{2345} - 3p\ 4P_{123}$ ;  $3d\ 4F_{2345} - 3p\ 4D_{1234}$ ;  $3d\ 4F_{2345} - 3p\ 4S_2$ .

	$3d$ $3p$	$4F_2$	145.5	$4F_3$	325.4	$4F_4$	527.7	$4F_5$
6.	$4P'_1$	0. 2891.48 34374.2		—	—	—	—	—
	182.6							
	$4P'_2$	3. 2876.43 dr. 34755.1	144.3	1. 2888.43 34610.8		—	—	—
	222.5							
	$4P'_3$	1. 2858.02 34979.0	145.4	2. 2869.95 34833.6	325.9	2. 2997.05 34507.7		—
7.	$4D_1$	2. 3190.89 31330.2		—	—	—	—	—
	144.1							
	$4D_2$	3. 3176.17 31475.8	145.6	2. 3190.88 31330.2		—	—	—
	249.3							
	$4D_3$	2. 3151.16 31725.2	145.7	2. 3165.70 31579.5	325.0	5. 3198.62 31254.5		—
8.	$4D_4$			2. 3132.22 31917.2	325.4	3. 3164.46 31591.8	527.7	8. 3218.22 31064.1
	$4S_2$	7. 3542.89 28217.5	145.4	4. 3561.24 28072.1		—	—	—

Argon II.  $4d\ 4F_{2345} - 4p\ 4P_{123}$ ;  $4d\ 4F_{2345} - 4p\ 4D_{1234}$ ;  $4d\ 4F_{2345} - 4p\ 4S_2$ .

	$4d$ $4p$	$4F_2$	266.33	$4F_3$	449.60	$4F_4$	531.66	$4F_5$
VI.	$4P_1$	2. 3263.582 30632.37		—	—	—	—	—
	357.30							
	$4P_2$	2. 3226.03 30988.94	266.33	( 3275.31) (30722.61)		—	—	—
	307.78							
	$4P_3$	1. 3194.270 31297.04	266.33	( 3221.70) (31030.71)	449.57	( 3269.04) (30581.14)		—
VII.	$4D_1$	6. 3581.652 27912.14		—	—	—	—	—
	260.31							
	$4D_2$	5. 3548.530 28172.66	266.33	6. 3582.397 27906.34		—	—	—
	494.40							
	$4D_3$	( 3487.30) (28666.99)	266.33	6. 3520.041 28400.66	449.55	8. 3576.658 27951.11		—
VIII.	$4D_4$			4. 3466.383 28840.29	449.63	5. 3521.281 28390.66	531.66	10. 3588.483 27859.00
	$4S_2$	2. 3952.742 25291.76	266.60	4. 3994.885 25025.16		—	—	—

TABLE 3. Neon II.

$\begin{pmatrix} 3d \\ 4s \end{pmatrix}?$ 3p	$g\ 4P_1$	305.5	$f\ 4P_2$	376.6	$e\ 4P_3$
$4P'_1$	2. 2770.60 36082.6	305.5	3. 2794.26 35777.1		—
182.6					
$4P'_2$	3. 2756.64 36265.4	305.6	2. 2780.06 35959.8	376.6	4. 2809.50 35583.1
222.6					
$4P'_3$	—		3. 2762.97 36182.3	376.9	5. 2792.05 35805.4
249.6					
$4D_1$	2. 3044.16 32840.2	305.5	1. 3072.70 32535.2		—
144.1					
$4D_2$	2. 3030.85 32984.5	305.3	3. 3059.16 32679.2	376.3	(3094.8) (32302.9)
337.2					
$4D_3$	—		3. 3035.98 32928.7	376.3	2. 3071.08 32552.4
439.40					
$4D_4$	—	—			3. 3039.65 32889.0
456.47					
$4S_2$	2. 3362.89 29727.8	305.9	1. 3397.86 29421.9	377.0	1. 3441.96 29044.9

## Argon II.

$\begin{pmatrix} 4d \\ 5s \end{pmatrix}?$ 4p	$g\ 4P_1$	729.50	$f\ 4P_2$	627.90	$e\ 4P_3$
$4P'_1$	3. 3669.550 27243.56	729.87	4. 3770.569 26513.69		—
357.30					
$4P'_2$	6. 3622.204 27599.69	728.96	6. 3720.467 26870.73	627.99	6. 3809.499 26242.74
307.78					
$4P'_3$	—		6. 3678.328 27178.54	627.84	8. 3765.313 26550.70
4076.704					
$4D_1$	8. 4076.704 24522.72	730.96	1. 4201.946 <sup>1)</sup> 23791.76		—
260.31					
$4D_2$	6. 4033.872 24783.10	729.04	6. 4156.135 24054.06	627.80	(4267.52) <sup>2)</sup> (23426.26)
494.40					
$4D_3$	—		5. 4072.429 24548.48	627.91	5. 4179.329 23920.57
439.40					
$4D_4$	—	—			10. 4103.957 24359.88
4564.37					
$4S_2$	3. 4564.37 21902.7	729.59	(4721.65) <sup>2)</sup> (21173.11)	627.90	6. 4865.96 20545.21

<sup>1)</sup> Coincidence.<sup>2)</sup> Note added to proof. These lines have been found on our plates.

Observed: 3. 4267.49; 23426.4. 3. 4721.66; 21173.1.

TABLE 4. Argon II  $3d\ 4D_{1234} - 4p\ 4P_{123}$ ;  $3d\ 4D_{1234} - 4p\ 4D_{1234}$ .

$3d$ $4p$	$4D'_1$	107.00	$4D'_2$	149.53	$4D'_3$	153.90	$4D'_4$
$4P'_1$	4. 4352.198 22970.46	106.91	3. 4332.035 23077.37		—		—
357.30							
$4P'_2$	1. 4420.943 22613.14	107.25	3. 4400.101 22720.39	149.51	4. 4371.334 22869.90		—
307.78							
$4P'_3$	—		2. 4460.512 22412.69	149.26	4. 4431.002 22561.95	153.86	5. 4400.986 22725.81
$4D_1$	2. 3891.400 25690.44	107.05	3. 3875.256 25797.49		—		—
260.31							
$4D_2$	2. 3931.232 25730.14	106.87	3. 3914.781 25537.01	149.61	4. 3891.978 25686.62		—
494.40							
$4D_3$	—		2. 3992.046 25042.75	149.57	4. 3968.346 25192.92	153.84	4. 3944.259 25346.16
439.47							
$4D_4$	—	—	—		2. 4038.816 24752.76	153.95	6. 4013.852 24906.71

TABLE 5.

Multiplet	$\frac{\nu_{Ne\ II}}{\nu_{Ar\ II}}$
Neon II $n=3$	
Argon II $n=4$	
$ns\ 4P - np\ 4P$	1.315
$ns\ 4P - np\ 4D$	1.301
$ns\ 4P - np\ 4S$	1.281
$np\ 4P - nd\ 4P$	1.151
$np\ 4D - nd\ 4D$	1.143
$nd\ 4F - np\ 4P$	1.122
$nd\ 4F - np\ 4S$	1.118
$nd\ 4F - np\ 4D$	1.115

TABLE 6.

Neon II			Argon II		Neon II; $n = 3$ Argon II; $n = 4$	
Exper. Term	Termvalue (relative)	Term- difference	Termvalue (relative)	Term- difference	Atom- configuration	Theor. term
$4D'_4$	—		101914.40	153.98	$(s^2 p^4). 3d$	$3d \ 4D_4$
$4D'_3$	—		101760.42	149.62		$3d \ 4D_3$
$4D'_2$	—		101610.80	107.03		$3d \ 4D_2$
$4D'_1$	—		101503.77			$3d \ 4D_1$
$4P_3$	112500.0	517.8	100000.00	844.40	$(s^2 p^4). ns$	$ns \ 4P_3$
$4P_2$	111982.2	299.1	99155.60	515.70		$ns \ 4P_2$
$4P_1$	111683.1		98639.90			$ns \ 4P_1$
$4P'_3$	85438.2	222.6	79198.25	307.75	$(s^2 p^4). np$	$np \ 4P_3$
$4P'_2$	85215.6	182.5	78890.50	357.30		$np \ 4P_2$
$4P'_1$	85033.1		78533.20			$np \ 4P_1$
$4D_4$	82522.2	337.3	77007.70	439.36		$np \ 4D_4$
$4D_3$	82184.9	249.7	76568.34	494.57		$np \ 4D_3$
$4D_2$	81935.2	144.1	76073.77	260.32		$np \ 4D_2$
$4D_1$	81791.1		75813.45			$np \ 4D_1$
$4S_2$	78678.0		73192.72			$np \ 4S_2$
$4D'_4$	52493.9	81.5	50566.20	121.80	$(s^2 p^4). nd$	$nd \ 4D_4$
$4D'_3$	52412.4	106.2	50444.40	188.61		$nd \ 4D_3$
$4D'_2$	52306.2	98.3	50255.79	107.79		$nd \ 4D_2$
$4D'_1$	52207.9		50148.00			$nd \ 4D_1$
$4F_5$	51458.6	528.1	49148.70	531.55	$(s^2 p^4). nd$	$nd \ 4F_5$
$4F_4$	50930.5	325.6	48617.15	449.59		$nd \ 4F_4$
$4F_3$	50604.9	145.5	48167.56	266.33		$nd \ 4F_3$
$4F_2$	50459.4		47901.23			$nd \ 4F_2$
$e \ 4P_3$	49633.0	376.7	52647.50	627.76	$(s^2 p^4). nd$	$nd \ 4P_3$
$f \ 4P_2$	49256.8	305.5	52019.74	729.14		$nd \ 4P_2$
$g \ 4P_1$	48950.8		51290.60			$nd \ 4P_1$

1) In  $Ne II$  probably:  $(s^2 p^4). 4s$ .

### 5. ZEEMAN-effects.

The ZEEMAN-effect of *Ne II* is never investigated. Now, we can calculate and predict the ZEEMAN-types from the classification and the theory of LANDÉ. In the case of *Ar II* the ZEEMAN-effect of some lines in the visible part of the spectrum is known by work of LÜTTIG. His results are only qualitative because he used a discharge perpendicular to the field. The magnetic field used by LÜTTIG was not very strong too. Now it will be very interesting to investigate the ZEEMAN-effect for checking the term-scheme especially in connexion with the *g*-values of the terms.

#### ZEEMAN-effect. Multiplet I.

Calculated<sup>1)</sup> ZEEMAN-effect from our classification, observed ZEEMAN-effect by LÜTTIG<sup>2)</sup>.

Int.	$\lambda$	Term-combination	ZEEMAN-effect
8.	5062.019	$4P_1 - 4P'_2$	Calc. (0.47) 1.27 2.20 Obs.: At 17000 Gausz the line splits up in four or six components. More components excluded.
8.	5009.246	$4P_2 - 4P'_3$	Calc.: (0.07) (0.20) 1.40 1.53 1.67 1.80 Obs.: Diffuse triplet
4.	4972.16	$4P_1 - 4P'_1$	Calc. (0) 2.66 Obs.: Triplet.
6.	4933.226	$4P_2 - 4P'_2$	Calc.: (0) 1.73 Obs.: Triplet. More components excluded.
8.	4847.783	$4P_2 - 4P'_1$	Calc.: (0.47) 1.27 2.20 Obs.: Quartet. Probably more components. Even type.
10.	4805.893	$4P_3 - 4P'_3$	Calc.: (0) 1.60 Obs.: (0) 1.60 ( $H = \pm 6000$ )
6.	4735.885	$4P_3 - 4P'_2$	Calc.: (0.07) (0.20) 1.40 1.53 1.67 1.80 Obs.: (0) 1.48 ( $H = \pm 7000$ ).

### 6. Conclusion.

A comparison has been made between the deep quarter terms of ionized *Neon* (*Ne II*) and ionized *Argon* (*Ar II*). Nearly all the terms predicted by the theory of HEISENBERG and HUND have been detected and identified.

In conclusion the author wishes to express thanks to Prof. P. ZEEMAN for the valuable advise and helpful suggestions.

*Laboratory "Physica" of the University of Amsterdam.*

March 1928.

<sup>1)</sup> The ZEEMAN-types have been calculated with the normal *g*-values. Anomalous *g*-values however can be present.

<sup>2)</sup> O. LÜTTIG: *Dissertatie. Halle a. S. 1911. Ann. d. Phys. 38, 69, 1912.*

**Anatomy.** — *Experimentell-anatomische Untersuchungen über die optischen Systeme im Gehirn.* By B. BROUWER, G. J. VAN HEUVEN and A. BIEMOND.

(Communicated at the meeting of June 30, 1928)

In Zusammenwirkung mit Prof. W. P. C. ZEEMAN habe ich vor einigen Jahren experimentell-anatomische Untersuchungen über die Frage begonnen, wie die verschiedenen Teile der Netzhaut im Zentralnervensystem vertreten sind. Die Methode war folgende. ZEEMAN laederte die Retina von Kaninchen, Katzen und Affen an circumscripten Stellen und zerstörte dadurch die Schicht der Ganglionzellen. Die Tiere wurden nach 18 Tagen getötet. Das Zentralnervensystem wurde nach der MARCHI-methode bearbeitet. In dieser Weise konnten die sekundär degenerierten Fasern in den Nervi optici, dem Chiasma, den Tractus optici und in den primären optischen Endigungsstätten studiert werden.

Die Verhältnisse bei Kaninchen und Katzen sind genau ausgearbeitet und beschrieben von Dr. J. F. A. OVERBOSCH. Es hat sich herausgestellt, dass die verschiedenen Retinaquadranten ihren bestimmten Platz im Corpus geniculatum externum besitzen und — so weit es das Kaninchen betrifft — auch im Corpus quadrigeminum anticum. Die Ergebnisse der Untersuchungen im ersten optischen Neuron beim Affen sind schon von ZEEMAN und mir an anderer Stelle in die Literatur gebracht. Ich beschränke mich darauf, daran zu erinnern, dass die Ganglionzellen der oberen Netzhaustranten ihre Achsenzylinder nach dem medialen Teil des Corpus geniculatum externum schicken, die unteren Quadranten nach dem lateralen. Die Macula hat eine ausgedehnte Projection in diesem Ganglion, was nicht zu verwundern ist, weil dieses Areal der Netzhaut sehr reichlich mit Ganglionzellen versehen ist; sie schickt ihre Fasern nach dem mittleren und zum grössten Teil nach dem dorsalen Gebiet des Geniculatums. Teilweise drängt also das Maculafeld die dorsalen und ventralen peripheren (peripher = nicht-maculär) Netzhaustranten auseinander. Ein Teil dieser letzteren muss jedoch im ganzen ventralen Gebiete vergegenwärtigt sein. Man vergleiche die Figur 1, aus welcher man ersehen kann, wie ZEEMAN und ich uns die Lokalisation der verschiedenen Netzhautteile im Corpus geniculatum externum beim Affen denken. Es können genügende Argumente für die Auffassung angeführt worden, dass analoge Verhältnisse beim Menschen bestehen.

Auf zwei Umstände aus dieser Untersuchungsreihe möchte ich hier noch besonders hinweisen. Erstens, dass derjenige Teil der Retina, den die

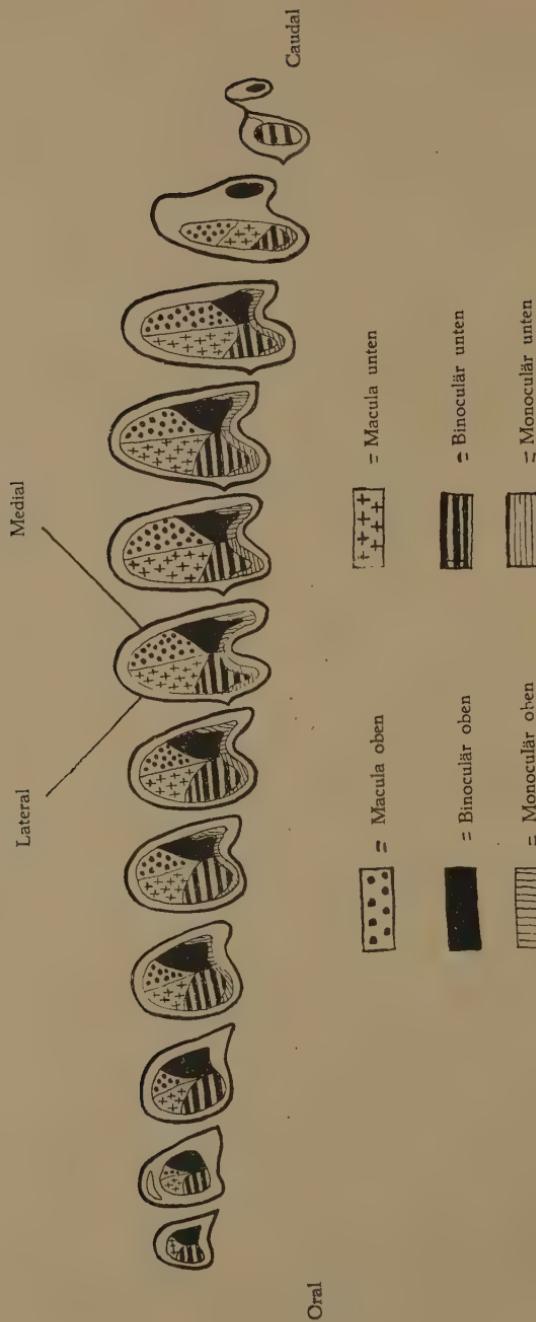


Fig. 1. Die Projection der verschiedenen Retinateile im Corpus geniculatum externum beim Affen.

Reize aus dem monoculären Gebiet des Gesichtsfeldes treffen, auch seinen besonderen Platz im Corpus geniculatum externum besitzt und zwar an dessen ventralem Rande. Zweitens, dass es uns, nicht gelungen ist, eine mehr detaillierte Lokalisation in den Unterteilen eines Quadranten aufzufinden. Im Gegenteil, wir wurden getroffen durch die Tatsache, dass die Degenerationsareale, die durch Laesionen verschiedener Stellen eines Quadranten hervorgerufen waren, so viel Uebereinstimmung miteinander zeigten. Dieser Befund wurde auch von OVERBOSCH bei Kaninchen und Katzen erhoben. In unseren Praeparaten konnte ich leicht ablesen, ob ZEEMAN eine Laesion temporal-oben, temporal-unten, nasal-oben oder nasal-unten gemacht hatte. Mit Ausnahme des monoculären Teiles jedoch konnte man im Corpus geniculatum externum nicht sehen, ob die Laesion in der Nähe der Macula oder an einer mehr peripheren Stelle gemacht worden war. Auf die Bedeutung dieser beiden Tatsachen werde ich unten weiter zurückkommen.

Da diese secundäre Degeneration, mit der MARCHI-methode bearbeitet, die Grenzen eines laedierten Neurones nicht überschreitet, waren wir in dieser Weise nicht in der Lage, die Projection der Retina auf die Grosshirnrinde festzustellen. Um dieses zu erreichen, wurden umschriebene Laesionen in der Area striata gemacht, wodurch eine retrograde Degeneration im Corpus geniculatum externum hervorgerufen wird. In Zusammenwirkung mit dem Augenarzt G. J. VAN HEUVEN haben wir solche Experimente bei neun erwachsenen Affen (*Cynomolgus fascicularis*) gemacht. Wir haben die Tiere nach der Operation eine ziemlich lange Zeit leben lassen (gewöhnlich 8 Monate), damit die retrograde Degeneration eine intensive würde. Die Gehirne wurden in Schnittserien zerlegt und abwechselnd gefärbt nach der VAN GIESON- und der WEIGERT-PALMethode. Diese beiden Formen von Degeneration, die retrograde von der Regio calcarina aus hervorgerufen und die secundäre MARCHI-degeneration von der Netzhaut aus, begegneten einander in dieser Weise. Wir konnten uns jetzt eine Vorstellung über die Frage bilden, wie die optischen Fasern sich im zweiten Neuron verbreiten.

Die Ausdehnung der Laesionen im Occipitalhirn und die dadurch verursachten degenerativen Veränderungen im Fasciculus longitudinalis inferior und im Corpus geniculatum externum wurden von Dr. VAN HEUVEN in diesen neun Schnittserien gezeichnet. Eine detaillierte Beschreibung dieser Ergebnisse wird in seiner Dissertation erscheinen. Einige der wichtigsten Resultate mögen hier folgen.

Es hat sich herausgestellt, dass die sogenannte Sehstrahlung in einigen Fällen bei der Operation mitlaediert worden war, in anderen jedoch nicht. Je nach der Lage und der Ausdehnung des primären Herdes in der Area striata gehen Zellen im Corpus geniculatum externum — und zwar ausschliesslich im gleichseitigen — zu Grunde. Dabei tritt eine reactive Glia-wucherung auf. Dieser Zellausfall tritt nur dann auf, wenn auch die tieferen Schichten des Feldes 17 von BRODMANN laediert worden sind (Figuren 2

Laesion in der  
Area striata



Fig. 2. Kleine Laesion in der Area striata des Affen.  
(Nach einem VAN GIESON-Praeparat).

— Laesion in der  
Area striata



Fig. 3. Grosse Laesion im dorso-medialen Teil der Area striata des Affen.  
(Nach einem WEIGERT-PAL-Praeparat).

und 3). Die Figuren 4 und 5 mögen zeigen, wie scharf eine bestimmte Laesion in der Occipitalrinde sich in den Zellen dieses Ganglions ausdrückt.

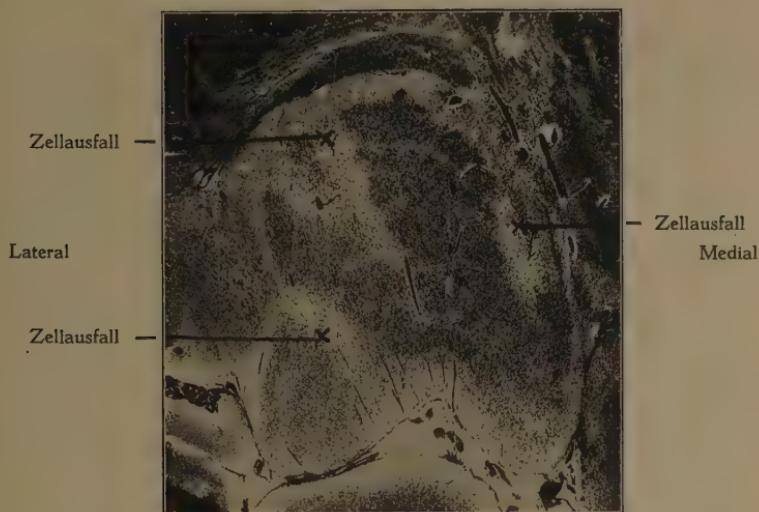


Fig. 4. Drei umschriebene Bezirke von retrograder Degeneration im Corpus geniculatum externum des Affen.  
(Nach einem VAN GIESON-Praeparat).

Die Untersuchungen lehrten, dass eine Laesion desjenigen Teiles der Area striata, der beim Affen auf der lateralen Hirnoberfläche gelegen ist, einen Zellausfall im mittleren und zum Teil auch im dorsalen Areal des Corpus geniculatum externum hervorruft. Da die Untersuchungen von ZEEMAN und mir gelehrt hatten, dass hier das Maculafeld localisiert werden musste, liess sich daraus schliessen, dass dieser laterale Teil der Area striata ausschliesslich als eine Endigungsstätte von maculären Verbindungen betrachtet werden muss. Es war jedoch möglich, noch eine weitere Unter-Verteilung zu machen. Das obere Areal der lateralen occipitalen Hirnoberfläche gehört dem medialen, das untere dem lateralen Gebiete der maculären Zône im Corpus geniculatum externum an. Es ist jedoch sicher, dass hiermit das Projectionsgebiet der maculären Verbindungen nicht erschöpft ist. Ein Teil davon greift auch auf die mediale Hirnoberfläche über, während mehrere Argumente für die Auffassung sprechen, dass auch die Windungen, die in der Tiefe um den Sulcus calcarinus herum gelegen sind, maculäre Verbindungen besitzen.

Die verschiedenen Tatsachen stimmen am besten miteinander überein, wenn man annimmt, dass eine sehr bestimmte topographische Beziehung zwischen dem Corpus geniculatum externum und der Regio calcarina besteht. Was oben in der Area striata liegt, findet sich medial im Corpus

geniculatum externum, dessen laterale Partie dem entspricht, was unten in der Area striata liegt. Ausserdem stehen die oralen Teile des Ganglions



Fig. 5. Zellausfall im Corpus geniculatum externum des Affen nach einer Laesio in der Area striata.  
(Nach einem VAN GIESON-Praeparat).

mit den mehr nach vorne gelegenen in der Regio calcarina in Verbindung, die caudalen jedoch mehr mit ihren hinteren Partieen. Leider ist es uns nicht gelungen, die am meisten nach vorne gelegenen Gebiete der Area striata zu extirpieren, sodass die retrograde Degeneration in keiner unserer Schnittserien in den meist oralen Abschnitten des Geniculatums gefunden wurde.

Ich möchte hervorheben, dass der Ausfall von Zellen sich häufig bis in den ventralen Rand fortsetzt. Die Bedeutung dieser Tatsache ist folgende. Im Corpus geniculatum externum finden sich normalerweise über einen ziemlich grossen Abschnitt verteilt am ventralen Rande einige Reihen von Zellen, die grösser als die übrigen dieses Ganglions sind. Dieser Kranz von



Fig. 6. Corpus geniculatum externum des Affen. Retrograde Degeneration kleinere und grösserer Zellen.  
(Nach einem VAN GIESON-Präparat).

grösseren Zellen, der beim Menschen noch stärker ausgesprochen ist als beim Affen, nimmt in der Literatur über das zerebrale optische System einen besonderen Platz ein. Beim Studium von Herden im Occipitalhirn des Menschen ist wiederholt festgestellt worden, dass diese grösseren Zellen wenig oder überhaupt nicht an der retrograden Degeneration mitbeteiligt waren. Dieser Umstand hat oft den Gedanken aufkommen lassen, dass dieser Teil des Corpus geniculatum externum nicht auf die Area striata projiziert ist. Einige nehmen an, dass wahrscheinlich hieraus Verbindungen mit dem Mittelhirn entstehen. Weil wir an zahlreichen Stellen in unseren Schnittserien beobachtet haben, dass die grösseren Zellen zugleich mit den kleineren zu Grunde gegangen waren, wird man verstehen, dass wir die Ueberzeugung gewonnen haben, dass alle Zellen dieses Ganglions ihre Achsenzylinder nach der occipitalen Rinde schicken. Die Figur N°. 6 zeigt einen solchen Zellausfall in deutlicher Weise. Ich werde unten auf die Bedeutung dieser Tatsache für die Physiologie zurückkommen.

Es ist nicht leicht, die Verhältnisse im Fasciculus longitudinalis inferior

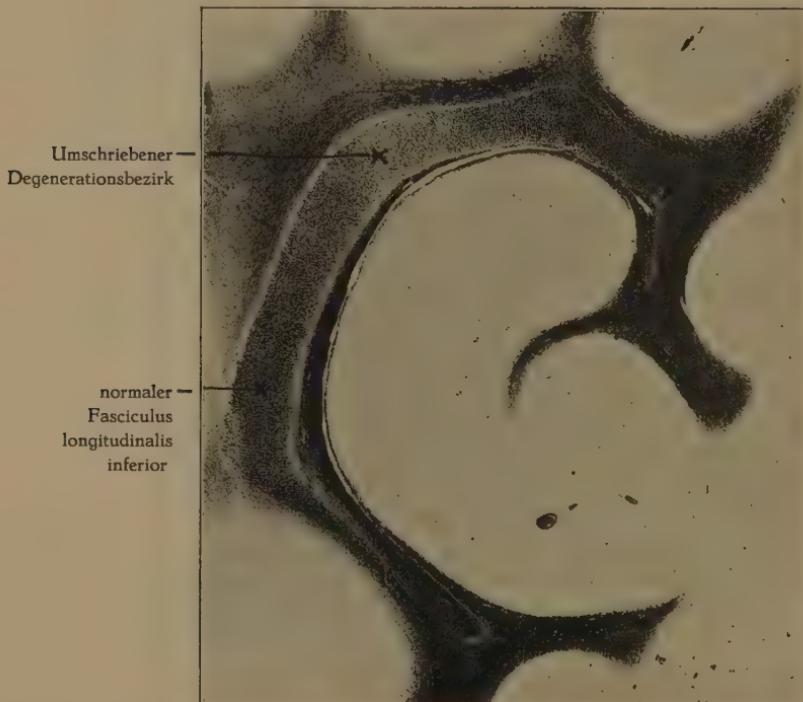


Fig. 7. Umschriebene Degeneration im Fasciculus longitudinalis inferior nach einer Lesion in der Area striata.  
(Nach einem WEIGERT-PAL-Praeparat).

zu übersehen. Doch liess sich in verschiedenen Schnittserien feststellen, dass eine gewisse Lokalisation der verschiedenen Netzhautteile in der Sehstrahlung bestehen muss. Dieses ist vor allem im Occipitalhirn deutlich, also in der Nähe der primären Herde. Die Figur N°. 7 gibt ein Beispiel einer ziemlich scharf umschriebenen Degeneration im Stratum sagittale externum. Die Fasern, welche mit den oberen Calcarinagebieten in Verbindung stehen, liegen in der Sehstrahlung des Affen dorsal, die, welche den mehr nach unten gelegenen Windungen angehören, finden sich in dieser Strahlung ventral. Die Fasersysteme, die sich nach der Rinde der lateralen Hirnwindungen wenden, werden im Fasciculus longitudinalis inferior mehr im mittleren Teil gefunden und nehmen ein grosses Gebiet dieser Strahlung ein. Wenn man die Schnittserien in mehr oraler Richtung verfolgt, so wird eine solche Lokalisation viel weniger deutlich. Zum Teil ist dieses dem Umstand zu zuschreiben, dass im Fasciculus longitudinalis inferior viele associativen Verbindungen aus anderen Hirnwindungen verlaufen. Die Degeneration der Fasern ist jedoch im Corpus geniculatum externum selbst oft wieder ziemlich scharf localisiert, in Uebereinstimmung mit dem umschriebenen Ausfall von Zellen.

In den Figuren N°. 8, 9 und 10 haben wir wiedergegeben, zu welchen Schlussfolgerungen man auf Grund der verschiedenen Tatsachen berechtigt ist. Ein sehr grosser Teil der Area striata wird von dem maculären Gebiet eingenommen. Die dorsalen Quadranten liegen auf der Rinde dorsal von den ventralen. Die Teile der Netzhaut, welche nicht zur Macula gehören,

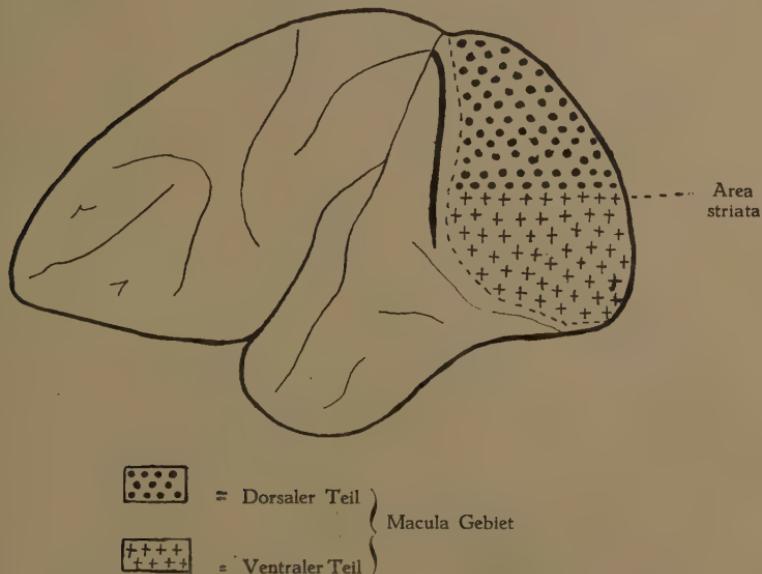


Fig. 8. Projection vom Macula-Gebiet auf der lateralen Hirnoberfläche beim Affen.

finden ihre Repräsentation in den Windungen rings um den Sulcus calcarinus. Auch hier ist wieder festzustellen, dass die oberen Quadranten dorsal

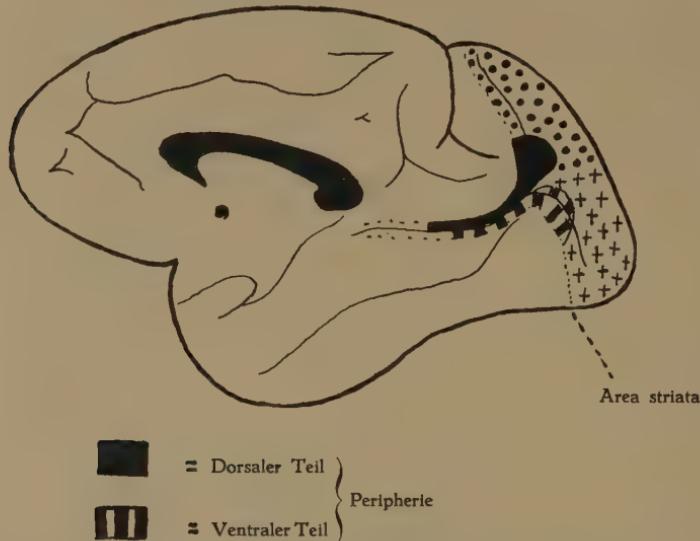


Fig. 9. Projektion von den verschiedenen Retina-Teilen auf der medialen Hirnoberfläche beim Affen.

und die unteren ventral liegen (Figur 10). Wir müssen jedoch aus unseren Schnittserien schliessen, dass auch maculäre Teile in den Windungen rings um den Sulcus calcarinus liegen, und dass diese zwischen den dorsalen und ventralen peripheren Retinaquadranten gesucht werden müssen.

Wie oben schon erwähnt wurde, ist bei unseren Operationen das meist nach vorne gelegene Gebiet der Area striata verschont worden, und sind auch die meist oral gelegenen Abschnitte des Corpus geniculatum externum von der Degeneration freibleiben. Weil alles auf eine ziemlich schematische Beziehung zwischen der ersten und zweiten optischen Station hinweist, schliessen wir daraus, dass in diesem oralen Gebiete der Area striata die verschiedenen Retinateile wieder vergegenwärtigt sind. In dieser Weise kann man dann die Projektion der Retina auf die Grosshirnrinde durch das Glasmodell wiedergeben, das ich jetzt demonstriere und das mehr genau von Dr. VAN HEUVEN in seinem Buch erklärt werden wird. Die Verhältnisse in der Area striata werden schon vorbereitet in Fasciculus longitudinalis inferior, in dem die maculären Verbindungen sich über eine grosse Ausdehnung zerstreuen und die Neigung zeigen, die dorsalen und ventralen peripheren Retinaquadranten auseinander zu drängen.

Auch bei diesen Untersuchungen im zweiten optischen Neuron drängte sich das Lokalisationsprinzip, von HENSCHEN immer verteidigt, uns auf. Zu dieser Schlussfolgerung war auch PUTNAM gekommen, der die corticalen

Verhältnisse beim Kaninchen in unserem Laboratorium ausgearbeitet hat. Dass sich jedoch erhebliche Differenzen mit der Lehre HENSCHEN's bei unseren Untersuchungen ergaben, ist nicht weniger deutlich. Diese Differenzen können nicht erklärt werden aus dem Umstand, dass diese Theorie aufgestellt worden ist für den Bau des optischen Systemes des Menschen, während wir bei höheren Säugetieren arbeiteten. Sie betreffen prinzipielle Punkte.

In der Vorstellung HENSCHEN's ist die Calcarinazône als eine anato-

MACULA OBEN      MACULA UNTEN.  
PERIPHERIE OBEN      PERIPHERIE UNTEN.

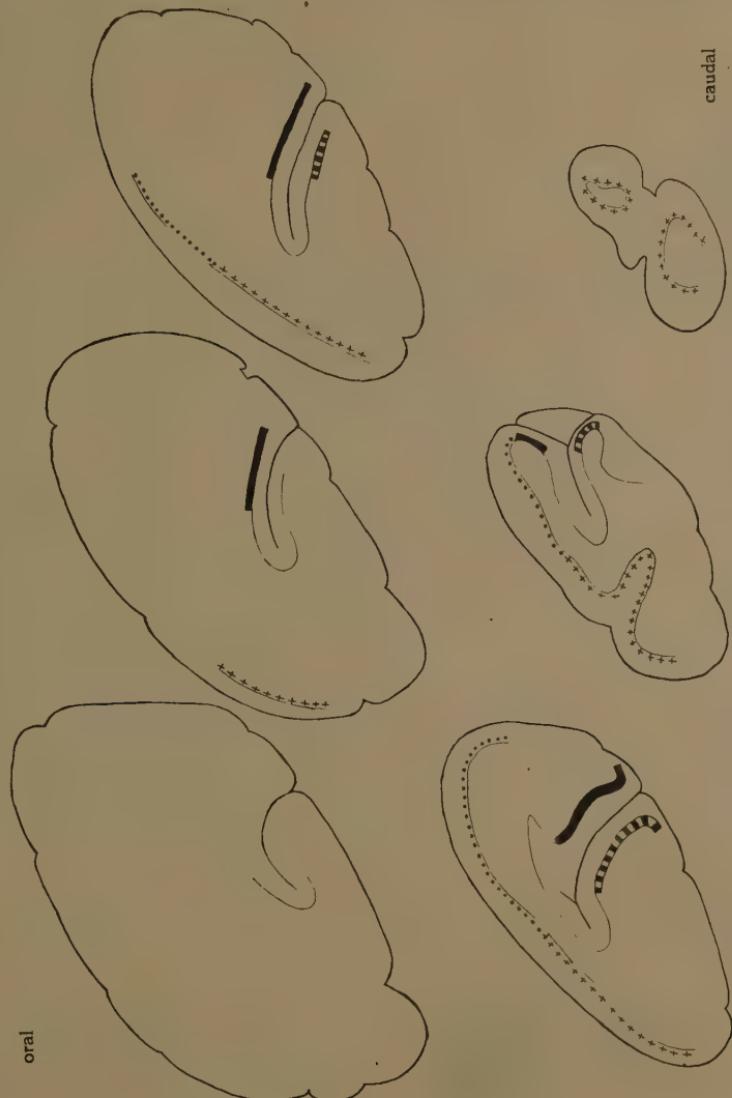


Fig. 10. Experimentell-anatomische Tatsachen über die Projection der verschiedenen Retinatellen beim Affen auf der Area striata.

mische Abspiegelung der Retina zu betrachten. Erstens muss darauf hingewiesen werden, dass man von einer eilandförmigen Repraesentation der Macula auf den Cortex, wie dieser Untersucher annimmt, nach unserer Erfahrung nicht sprechen darf. Die maculären Verbindungen haben eine lokalisierte, aber eine sehr ausgedehnte Endigungsstätte auf der occipitalen Rinde. Aber auch was die übrigen Teile der Netzhaut betrifft, sind wir zu anderen Resultaten gekommen. Wohl sind auch beim Affen die Verhältnisse so, dass die oberen Retinaquadranten dorsal und die unteren ventral liegen; jedoch wurde oben schon darauf hingewiesen, dass es uns bei unseren Untersuchungen im ersten optischen Neuron nicht möglich gewesen ist, eine Lokalisation innerhalb eines Quadrantes aufzudecken. Weil nun die anatomischen Beziehungen zwischen dem Corpus geniculatum externum und der Area striata ziemlich schematische lokalisierte sind, ist es deutlich, dass wir nicht genügend Gründe haben mit HENSCHEN anzunehmen, dass jeder Punkt der Retina seinen umschriebenen Platz in der Occipitalrinde besitzt. Dieses negative Ergebnis bekommt grösseren Wert, wenn wir den folgenden Punkt betrachten. Dadurch dass es ZEEMAN gelungen ist, isolierte Laesonen sehr weit nasal in der Netzhaut zu machen, wurde es uns möglich, die Projektion des Retinateiles, der ausschliesslich Reize aus dem monokulären Gesichtsfelde aufnehmen muss, im Zentralnervensystem zu studieren. Wir waren zu der Schlussfolgerung gekommen, dass dieser Teil der Netzhaut im gekreuzten Corpus geniculatum externum tatsächlich seinen absonderlichen Platz besitzt und zwar am ventralen peripheren Rande (man vergleiche die Figur N°. 1). Dr. VAN HEUVEN und ich erwarteten nun, dass auch dieser Teil seine eigene Projection auf die Area striata besitzen würde. Obschon wir durch unsere Experimente in der occipitalen Zône wiederholt Zellausfall im monokulären Gebiete des Corpus geniculatum externum erzeugt haben, ist es uns nicht gelungen, die Ergebnisse derart zu verwerten, dass hieraus die Existenz einer absonderlichen monokulären Zône im der Area striata abgeleitet werden konnte. Wir haben dadurch die Ueberzeugung gewonnen, dass es wohl eine topographische Lokalisation zwischen den verschiedenen Teilen des optischen Systemes: Retina, Corpus geniculatum externum, Fasciculus longitudinalis inferior und Area striata gibt, dass es aber nur eine Lokalisation bis zu einer gewissen Höhe ist. Es gibt keinen einfachen „Abklatsch“ der Retina im Gehirn, es existiert keine mathematische Projection, wie es HENSCHEN und Viele mit ihm annehmen. Mit anderen Worten: die Tatsache, dass das Individuum imstande ist, Objecte in der Aussenwelt mittelst optischer Reize ganz genau zu lokalisieren, verläuft nicht parallel — und findet also nicht ihre Erklärung — in einer scharfen Lokalisation der optischen Leitungsbahnen und Zentren im Zentralnervensystem.

Damit wir die Ursache dieser Gegenstellung beleuchten können, erwähnen wir jetzt einige Resultate von Untersuchungen über das folgende, also über das dritte optische Neuron. Wir haben uns die Frage vorgelegt:

wie verlaufen die optischen Systeme von der Area striata aus in die übrigen Teile des Gehirnes? In Zusammenwirkung mit meinem Assistenten A. BIEMOND sind umschriebene Laesonen in der Occipitalrinde des Affen gemacht worden. Die Tiere wurden noch 18 Tagen getötet und das Zentralnervensystem wurde nach der MARCHI-methode verarbeitet. Dr. BIEMOND wird später über die Resultate dieser Untersuchungen genau berichten. Es ist schon längere Zeit bekannt, dass ein Teil der cortico-fugalen Systeme sich über die angrenzenden Rindenareale ausbreitet, während ein anderer Teil sich nach niederen Niveaux im Zentralnervensystem wendet, insbesondere nach dem Corpus quadrigeminum anticum. Wir konnten jedoch feststellen, dass im Felde 17 von BRODMANN eine wichtige cortico-fugale Bahn entspringt, die im Corpus geniculatum externum derselben Seite ihr Ende findet. Diese Verbindung verläuft in den Strata sagittalia externa und interna. Die Figur N°. 10 zeigt ein Beispiel einer Laesion in der Area striata, Figur N°. 11 die secundäre Degeneration



Fig. 11. Primärer Herd in der Area striata des Affen.  
(Nach einem MARCHI-Präparat).

im Fasciculus longitudinalis inferior, Figur N°. 12 die Degeneration im Corpus geniculatum externum. Diese Verbindung zeigt eine unverkennbare Neigung zu einer bestimmten localisierten Endigung in diesem Ganglion. Nach Zerstörung der lateralen Oberfläche des Occipitalhirnes, die wir als eine Endigungsstätte von maculären Verbindungen kennen gelernt haben, findet man die Osmiumkörper hauptsächlich im mittleren Teil des Corpus

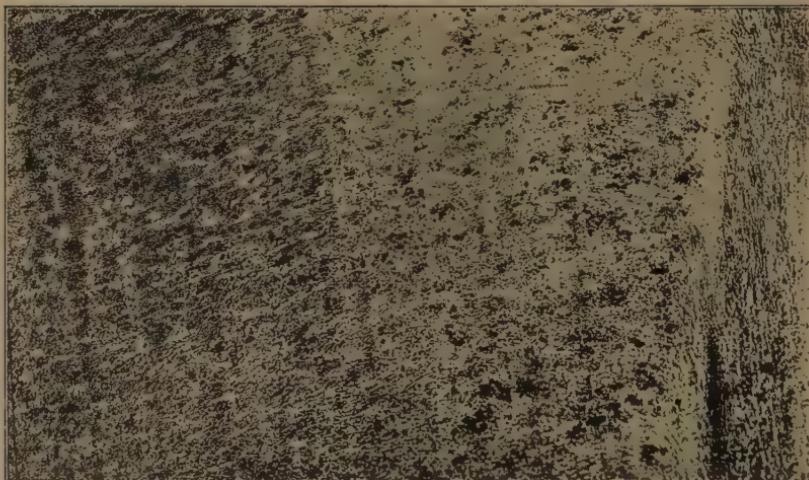


Fig. 12. Cortico-fugale Degeneration im Fasciculus longitudinalis inferior des Affen  
nach einer Laesion in der Area striata.  
(Nach einem MARCHI-Praeparat).



Fig. 13. Cortico-fugale Degeneration im Corpus geniculatum externum  
des Affen nach einer Laesion in der Area striata.  
(Nach einem MARCHI-Praeparat).

geniculatum externum. Man vergleiche in dieser Beziehung die Figur N°. 14 mit der Figur N°. 1. Im Fasciculus longitudinalis inferior verlaufen also nicht nur Erregungen vom Corpus geniculatum externum nach der Area striata, sondern auch in umgekehrter Richtung.

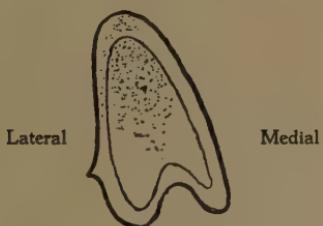


Fig. 14. Cortico-fugale Degeneration im Corpus geniculatum externum des Affen nach einer Verletzung im Occipitalhirn.  
(Nach einem MARCHI-Praeparat).

Dass im Occipitalhirn zentrifugale Fasern absteigen, die im Corpus geniculatum externum blind endigen, ist 1902 von M. PROBST bei Hunden und Katzen nachgewiesen worden. Seine Resultate sind jedoch nicht genügend in die Literatur gedrungen und ihnen ist auch widersprochen worden. Auf Grund unserer Untersuchungen an mehreren Schnittserien von Kaninchen und Affen müssen wir aber an der Existenz solcher Verbindungen festhalten. Die Frage, ob solche Systeme ausschliesslich in der Area striata entspringen, ist bis jetzt noch nicht genügend ausgearbeitet worden: ich habe jedoch Gründe, daran zu zweifeln.

Es ist deutlich, dass die Existenz von cortico-fugalen Verbindungen mit dem Corpus geniculatum externum ihre Bedeutung für die Physiologie haben muss. Das höchste Assoziationszentrum — die Grosshirnrinde — ist dadurch im Stande einen Einfluss auf das Geschehen in der primären optischen Hauptstation auszuüben. Dieser Einfluss darf nicht gleichgestellt werden mit dem, den die Grosshirnrinde auf niedere Reflexzentren der verschiedenen anderen Teile des Nervensystems ausübt. Denn das Corpus geniculatum externum darf nicht als ein Zentrum betrachtet werden, in dem sich niedere optische Reflexe abspielen können. Für ein derartiges Reflexzentrum ist die Anwesenheit von Zellen notwendig, die die Erregungen nach effektorischen Zellen abfliessen lassen. Wir haben nun oben gesehen, dass nach unserer Ueberzeugung alle Zellen des Corpus geniculatum externum ihre Achsenzylinder nach der occipitalen Rinde schicken; Die niederen optischen Reflexe müssen sich also ausserhalb dieses Ganglions im Mittelhirn abspielen. Aus der Existenz dieser cortico-fugalen Verbindungen müssen wir schliessen, dass hier anatomisch die Möglichkeit einer zentrifugalen Beeinflussung von rein zentripetalen Systemen gegeben ist.

Der Gedanke drängt sich auf, dass wir diesem zentrifugalen System die Funktion einer Hemmung, einer Inhibition zuschreiben müssen. Beim optischen System ist die Annahme einer solchen inhibierenden Funktion nur dann befriedigend, wenn man den Begriff „Hemmung“ oder „Inhibition“ von einem übergeordneten Standpunkt aus betrachtet. Es scheint mir sehr wahrscheinlich, dass das Individuum mittels seiner Grosshirnrinde in dieser Weise im Stande ist, die Erregbarkeit von Zellen im Corpus geniculatum externum zu vermindern, von anderen Zellen dagegen zu erhöhen. Die dominierende Bedeutung, die wir der Grosshirnrinde für die Aktivität im Zentralnervensystem zuerkennen müssen, lässt sich ganz gut vereinigen mit dem Gedanken, dass in dieser Weise ein führender, Richtung gebender Einfluss ausgeübt wird auf die optischen Erregungen, unmittelbar nachdem sie in diese erste Station des Gehirnes eingetreten sind.

Wir nähern uns hier der Psychologie. In seiner Arbeit über die Tätigkeit des Bewusstseins beim Entstehen und Fortbestehen der Reflexe hat M. STRAUB darauf hingewiesen, dass wir psychologische Begriffe bei den niederen Reflexen nicht entbehren können. Er knüpft an Auseinandersetzungen an, die SHERRINGTON bei seiner Besprechung der bedingten Reflexe PAWLOW's gegeben hat. Beim Einüben solcher Reflexe ist die Inhibition ein Factor von Bedeutung. Der erste Schritt ist das Lenken der Aufmerksamkeit auf die richtigen Reize. „Beim psychologischen Process, den wir Aufmerksamkeit nennen“, schreibt SHERRINGTON, „gibt es nebst dem Nachdruck, der auf die Reize gelegt wird, auf die Acht gegeben wird, eine gleichzeitige Hemmung anderer Reize, die auf anderen Wegen Zugang suchen zum Geist. Dieses Zurückhalten von Beantwortung anderer Reize ist ein unverkennbares Element im Process der Aufmerksamkeit.“

Ueberlegt man nun, wie HENSCHEN und seine Mitarbeiter zu der streng durchgeführten Lokalisationstheorie gekommen sind, so stellt sich folgendes heraus. Man studiert Patienten mit Defecten im Gesichtsfeld und bestimmt post mortem die Art, die Lage und die Ausdehnung der anatomischen Laesion und deduziert aus diesen beiden Gruppen von Tatsachen eine Schlussfolgerung über die Lokalisation von Funktionsstörungen im optischen System. Diese Methode liegt nicht ausschliesslich auf dem Gebiete der Anatomie und der Physiologie, sondern auch auf dem der Psychologie. Der Kliniker bedarf dabei der Mitarbeit des Bewusstseins des Patienten in intensiver Weise. Den Anteil der psychischen Aktivität des Untersuchten in den gefundenen Erscheinungen kann man nur bis zu einer gewissen Höhe bestimmen und nur in groben Zügen analysieren. Damit man das Geschehen bei dieser Arbeitsmethode besser durchschauen kann ist es vor allem notwendig, mehr genaue Erfahrungen zu sammeln über die oben beschriebenen cortico-fugalen Verbindungen und weiter über die associativen Relationen zwischen der occipitalen Rinde und den übrigen Territorien der Grosshirnrinde. Solche Untersuchungen sind jetzt in meinem Laboratorium im Gang. Darüber hoffe ich später zu berichten.

## LITERATUR.

1. B. BROUWER and W. P. C. ZEEMAN: Experimental anatomical Investigations concerning the projection of the retina on the primary optic centres in apes. The Journal of Neurology and Psychopathology, Vol. 6, 1925.
2. B. BROUWER and W. P. C. ZEEMAN: The Projection of the Retina in the Primary Optic neuron in Monkeys. Brain, Vol. 49, 1926.
3. S. E. HENSCHEN: On the Value of the Discovery of the Visual centre. Scandinavian Scientific Review, 3, 10, 1924.
4. S. E. HENSCHEN: Vierzigjähriger Kampf um das Sehzentrum und seine Bedeutung für die Hirnforschung. Zeitschrift für die gesammte Neurologie und Psychiatrie. 506, 87, 1925.
5. J. F. A. OVERBOSCH: Experimenteel-anatomische onderzoeken over de projectie der retina in het centrale zenuwstelsel. Dissertatie Amsterdam, 1927.
6. T. J. PUTNAM and I. KELLERS PUTNAM: The anatomic projection of the retinal quadrants on the striate cortex of the rabbit. The Archives of Neurology and Psychiatry, 1926.
7. M. PROBST: Ueber den Verlauf der centralen Sehfasern (Rinden-Sehhügelfasern) und deren Endigung im Zwischen- und Mittelhirne und über die Associations- und Commissurenfasern der Sehsphäre. Archiv für Psychiatrie und Neurologie, Band 35, 1902.
8. C. S. SHERRINGTON: The rôle of Reflexinhibition. Science Progress, №. 20, 1911.
9. M. STRAUB: De werkzaamheid van het bewustzijn bij het ontstaan en voortbestaan der reflexen. Geneeskundige Bladen. №. 7, 1915.

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**Mathematics.** — *Derivation of some Numbers relating to the System of Biquadratic Curves that pass through Five Points lying in an  $R_4$ .* By J. W. A. VAN KOL. (Communicated by Prof. HENDRIK DE VRIES.)

(Communicated at the meeting of March 31, 1928).

§ 1. Let us consider the biquadratic involutory transformation in a linear four-dimensional space  $R_4$  defined by the equations written in homogeneous coordinates:

$$x_i \cdot x_i = \varrho \quad (i = 1, \dots, 5).$$

Let  $A_1 A_2 A_3 A_4 A_5$  be the fundamental simplex.

Through this transformation to an angular point of the simplex there are associated  $\infty^3$  points, viz. the points of the space defined by the other four angular points. To any point of a side of the simplex there correspond  $\infty^2$  points, viz. the points of the "opposite" side plane. To any point of a side plane there correspond  $\infty^1$  points, viz. the points of the "opposite" side.

An arbitrary linear space is transformed into a biquadratic space  $\Omega^4$  containing the side planes of the simplex; the sides of the simplex are double lines, and the angular points are triple points.

An arbitrary plane is transformed into a surface of the sixth degree  $F^6$  containing the sides of the simplex and having triple points in the angular points.

An arbitrary line is transformed into a biquadratic (four-dimensional twisted) curve  $k^4$  passing through  $A_1, \dots, A_5$ . In this way we have defined a one-one representation of the curves  $k^4$  passing through  $A_1, \dots, A_5$  on the lines of  $R_4$ . This representation shows at once that *through seven given points there passes one curve  $k^4$* .

Through an arbitrary point there pass  $4 \cdot 6 - 5 \cdot 3 = 9$  lines that cut a curve  $k^4$  and a surface  $F^6$  in different points. Hence: *There are nine curves  $k^4$  that pass through six given points and cut a given plane and a given line.*

In the same way we can deduce: *There are three curves  $k^4$  that pass through six given points, cut a given plane and cut a given line passing through one of the given points outside that point.*

And: *There are five curves  $k^4$  that pass through six given points, cut a given line and cut a given plane passing through one of the given points outside that point.*

The curves  $k^4$  that pass through six given points and cut a given plane, form a space of the ninth degree  $\Omega^9$  that has sextuple points in the

given points; the given plane and the planes through three of the given points containing the curves  $k^4$  that are degenerate in two conics which cut each other once, are single planes in this space; each of the spaces through four of the given points contains a surface of the fifth degree formed by twisted cubics that are parts of curves  $k^4$  degenerate in a twisted cubic and a line which cut each other once.

The curves  $k^4$  that pass through six given points and cut a given line, form a surface of the ninth degree<sup>1)</sup> which has quadruple points in the given points; this surface contains the joins of the given points, the given line, and the twisted cubics that pass through four of the given points and cut the join of the remaining two given points and the given line.

The lines through an arbitrary point  $P$  that touch a space  $\Omega^4$ , form a "conical space" of the twelfth degree of which the lines  $PA_i$  ( $i = 1, \dots, 5$ ) are sextuple lines. This space is cut by a curve  $k^4$  outside the points  $A_i$  in  $4.12 - 5.6 = 18$  points. Hence:

*There are eighteen curves  $k^4$  that pass through six given points, cut a given line and touch a given linear space.*

In the same way we can show: *There are six curves  $k^4$  that pass through six given points, cut a given line through one of the given points outside that point, and touch a given linear space.*

The curves  $k^4$  that pass through six given points  $A_1, \dots, A_6$  and touch a given linear space  $R_3$ , form a space of the eighteenth degree that has twelvefold points in  $A_1, \dots, A_6$ ; the planes through three of the given points are double planes in this space; each of the spaces through four of the given points contains a surface of the tenth degree formed by twisted cubics that are parts of curves  $k^4$  which are degenerate in a twisted cubic and a line cutting each other once. The points of contact with  $R^3$  lie on a quadratic surface<sup>2)</sup>. For in the plane  $a \equiv A_1 A_2 A_3$ , we can indicate two such points, viz. the points where the line  $aR_3$  is touched by the conics passing through  $A_1, A_2, A_3$  and the point of intersection of  $a$  and the plane  $A_4 A_5 A_6$  and touching  $aR_3$ .

The transversals of two curves  $k^4_1$  and  $k^4_2$  that pass through  $A_1, \dots, A_5$ , form a surface  $\Omega^{11}$  of the degree eleven with sextuple points in  $A_1, \dots, A_5$ . The number of points of intersection of  $\Omega^{11}$  and an arbitrary line  $l$  is equal to the number of points where the axial projections of  $k^4_1$  and  $k^4_2$  out of  $l$  on an arbitrary plane cut each other outside the projections of  $A_1, \dots, A_5$ , hence equal to  $4^2 - 5 = 11$ . The number of points of intersection outside  $A_1$  of  $\Omega^{11}$  and a line  $m$  through  $A_1$  is equal to the number of points where the axial projections of  $k^4_1$  and  $k^4_2$  out of  $l$  on an arbitrary plane cut each other outside the axial projections of

<sup>1)</sup> Cf. Dr. G. SCHAAKE. A quadruple involution in space, these Proceedings 29, p. 811 in which paper also several other of the results found above are derived in another way.

<sup>2)</sup> This surface is identical with the quadratic surface  $\omega^2$  mentioned in the paper of Dr. SCHAAKE cited above, p. 808.

$A_2, \dots, A_5$ , hence equal to  $3^2 - 4 = 5$ .  $\Omega^{11}$  is cut by a third curve  $k^4_3$ , which passes through  $A_1, \dots, A_5$  outside these points in  $4 \cdot 11 - 5 \cdot 6 = 14$  points. Consequently:

*There are fourteen curves  $k^4$  that pass through five given points and cut three given lines.*

In a similar way we can show: *There are five curves  $k^4$  that pass through five given points, cut two given lines, and cut one given line through one of the given points outside that point. There are ten curves  $k^4$  that pass through five given points and cut three given lines two of which intersect each other, in different points.*

The curves  $k^4$  that pass through five given points and cut two given lines, form a space of the fourteenth degree with ninefold points in the given points; the given lines are quadruple lines and the planes through three of the given points are single planes in this space; each of the spaces through four of the given points contains two surfaces of the fifth degree formed by twisted cubics that are parts of degenerate curves  $k^4$ .

§ 2. The congruence of the curves  $k^4$  that pass through six given points  $A_1, \dots, A_6$  and cut a given plane  $\alpha$ , may be represented on  $\alpha$  by associating to each of the curves  $k^4$  of the congruence as image its point of intersection with  $\alpha$ . This representation has twenty singular points, viz. the points  $S_{ikl}$  where  $\alpha$  is cut by the planes  $A_i A_k A_l$  ( $i, k, l = 1, \dots, 6$ ;  $i, k, l$  unequal).  $S_{ikl}$  is the image of the  $\infty^1$  curves  $k^4$  of the congruence that are degenerate in two conics cutting each other of which one individual passes through  $S_{ikl}$ .

From a number deduced in § 1 it follows that the system of the curves  $k^4$  that cut besides a second given plane  $\varphi$ , is represented on a curve  $k_\varphi$  of the ninth order.  $k_\varphi$  passes through the points  $S_{ikl}$ . Two curves  $k_\varphi$  and  $k_\psi$  cut each other outside the singular points in 61 points.

Hence: *There are 61 curves  $k^4$  that pass through six given points and cut three given planes.*

In the same way we can deduce:

The system of the curves  $k^4$  that also touch a given linear space, is represented on a curve of the eighteenth order with double points in the points  $S_{ikl}$ .

*There are 122 curves  $k^4$  that pass through six given points, cut two given planes and touch a given linear space.*

*There are 244 curves  $k^4$  that pass through six points, cut a given plane and touch two given linear spaces.*

Let  $k$  be the image curve of the system of the curves  $k^4$  that pass through  $A_1, \dots, A_6$ , cut a twice and, accordingly, have two image points each. The order of  $k$  is evidently equal to the number of curves  $k^4$  that pass through  $A_1, \dots, A_6$ , cut a given line  $l$  and cut a given plane  $\alpha$  through  $l$  outside  $l$ ; hence it is also equal to the number of the points where

the surface  $F^9$  formed by the curves  $k^4$  that pass through  $A_1, \dots, A_6$  and cut  $l$ , is cut by  $a$  outside  $l$ . According to a theorem of PIERI<sup>1)</sup> the latter number is equal to the product of the degrees of  $F^9$  and  $a$ , diminished by the product of the multiplicities of  $l$  in  $F^9$  and  $a$ , and by the class of the envelope of the linear spaces that touch  $F^9$  and  $a$  in the same point of  $l$  and, accordingly, contain  $a$ .

As, according to § 1, the points of contact of the curves  $k^4$  that pass through  $A_1, \dots, A_6$  and touch a given linear space  $R_3$  through  $a$ , lie on a quadratic surface that is cut by  $l$  in two points, this class is equal to two. Consequently  $k$  is of the order  $9 - 1 - 2 = 6$ .  $k$  has single points in the points  $S_{ikl}$ . The intersection of  $k$  and the image curves found above gives:

*There are seventeen curves  $k^4$  that pass through six given points, cut one given plane twice and cut another given plane once.*

*There are 34 curves  $k^4$  that pass through six given points, cut a given plane twice and touch a given linear space.*

From the above we can also deduce properties of surfaces formed by systems of  $\infty^1$  curves  $k^4$ , such as:

The curves  $k^4$  that pass through six given points and cut a given plane twice, form a surface of the degree 17 that has sevenfold points in the given points and cuts the given plane along a curve of the sixth order; the joins of the given points are single lines of the surface; each plane through three of the given points is cut by the surface in three of these lines and a conic; each space through four of the given points is cut by the surface in six lines, four conics and a twisted cubic; the surface itself is a double surface of  $\Omega^9$  (§ 1).

The curves  $k^4$  that pass through six given points and cut two given planes, form a surface of the degree 61 that has 26-fold points in the given points and cuts the given planes in curves of the ninth order; the joins of the given points are quintuple lines of the surface; each plane through three of the given points is cut by the surface in three of these quintuple lines and two conics; each space through four of the given points is cut by the surface in six quintuple lines, eight conics and five twisted cubics.

§ 3. The congruence of the curves  $k^4$  that pass through five given points  $A_1, \dots, A_5$  and cut a given line  $a$  twice, may be represented on a plane  $a$  in the following way. We choose a conic  $k^2$  in  $a$  and we suppose a projective correspondence to be established between the points of  $a$  and those of  $k^2$ . To a curve  $k^4$  of the congruence that cuts  $a$  in  $A$  and  $A'$ , we shall now associate the point of intersection of the lines that touch  $k^2$  in the points that are associated to  $A$  and  $A'$ .

This representation does not contain any singular elements.

1) Cf. Rend. del Circolo Mat. di Palermo, t. 5, 1891.

It follows from numbers that have been deduced in § 1; that the system of the curves  $k^4$  of the congruence that cut a given plane, is represented on a curve of the third order and that the system of the curves  $k^4$  of the congruence that touch a given linear space, is represented on a curve of the sixth order. From this we can deduce the following numbers:

*There are nine curves  $k^4$  that pass through five given points, cut a given line twice and cut two given planes. There are eighteen curves  $k^4$  that pass through five given points, cut a given line twice, cut a given plane and touch a given linear space. There are thirty six curves  $k^4$  that pass through five given points, cut a given line twice and touch two given linear spaces.*

The curves  $k^4$  that pass through five given points, cut a given line twice and cut a given plane, form a surface of the ninth degree with triple points in the given points; the given line is a triple line of the surface; each space through four of the given points is cut by the surface in three twisted cubics; the given plane is cut by the surface in a conic. The latter property follows from the number:

*There are two curves  $k^4$  that pass through five given points, cut a given line once and cut another given line twice.*

This number may be deduced in the way indicated in § 1. But it follows also from the theorem that the curves  $k^4$  which pass through five given points and cut a given line twice, lie in the quadratic space that has the given line as double line and passes through the five given points.

§ 4. The congruence of the curves  $k^4$  that pass through five given points  $A_1, \dots, A_5$  and cut two given lines  $a_1$  and  $a_2$ , may be represented on a plane  $\alpha$  in the following way. We choose two points  $K$  and  $L$  in  $\alpha$ . We suppose a projective correspondence to be established between the points of  $a_1$  and the rays of the plane pencil  $(K, \alpha)$  and another one between the points of  $a_2$  and the rays of the plane pencil  $(L, \alpha)$ . To a curve  $k^4$  of the congruence which cuts  $a_1$  in  $P$  and  $a_2$  in  $Q$ , we associate the point of intersection of the rays that are associated to  $P$  and  $Q$ .

Let us call the points of  $a_1$  and  $a_2$  corresponding to the line  $KL$  that belongs to both plane pencils resp.  $A'_1$  and  $A'_2$ . The curve  $k^4$  of the congruence that passes through  $A'_1$  and  $A'_2$ , is a singular curve; it has any point of  $KL$  as image.

$K$  and  $L$  are singular points;  $K$  and  $L$  are resp. the images of the curves  $k^4$  of the congruence that pass through  $A'_2$  resp.  $A'_1$ .

The twisted cubic which passes through  $A_1, \dots, A_4$  and cuts  $a_1$  and  $a_2$ , is a part of  $\infty^1$  degenerate curves  $k^4$  that are represented in the same point  $S_5$ . In this way we find 5 singular points  $S_i$  ( $i = 1, \dots, 5$ ).

The conic that passes through  $A_1$  and  $A_2$ , cuts  $a_1$ ,  $a_2$  and cuts the

plane  $A_3 A_4 A_5$ , is a part of  $\infty^1$  degenerate curves  $k^4$  that are represented in the same point  $T_{12}$ . Thus we find 10 more singular points  $T_{ik}$  ( $i = 1, \dots, 5$ ;  $i \neq k$ ).

It follows from numbers that have already been found:

The system of the curves  $k^4$  of the congruence that cut a given plane, is represented on a curve of the order 18 with ninefold points in  $K$  and  $L$ , triple points in  $S_i$  and single points in  $T_{ik}$ .

The system of the curves  $k^4$  of the congruence that touch a given linear space, is represented in a curve of the order 36 with 18-fold points in  $K$  and  $L$ , sextuple points in  $S_i$  and double points in  $T_{ik}$ ,

This gives again the numbers:

*There are 107, 214, 428 curves  $k^4$  that pass through five given points, cut two given lines and besides resp. cut two given planes, cut one given plane and touch a given linear space, touch two given linear spaces.*

Finally we can again indicate properties of surfaces and spaces formed by systems of  $\infty^1$  resp.  $\infty^2$  curves  $k^4$ , as:

The curves  $k^4$  that pass through five given points, cut two given lines and cut a given plane, form a surface of the degree 107 with 45-fold points in the given points. The given lines and also the joins of the given points are ninefold lines of the surface. The latter results from the property that the twisted cubics which pass through three given points and cut three given lines, form a surface of the ninth degree<sup>1)</sup>. The surface has further a curve of the fourteenth order in common with the given plane, three ninefold lines and a conic with each of the planes through three of the given points, and six ninefold lines, four conics, one triple and twelve single twisted cubics with each of the spaces through four of the given points.

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<sup>1)</sup> Cf. Wiskundige opgaven, deel 14, opg. 146.

**Mathematics. — Two Representations of the Congruence of REYE.**  
By Prof. JAN DE VRIES.

(Communicated at the meeting of April 28, 1928).

§ 1. By the aid of a well known cubic transformation<sup>1)</sup> the twisted cubics  $k^3$  through four given points are transformed into the rays of space. In this case the congruence  $\Gamma'$  of the  $k^3$  through five given points (congruence of REYE) is represented on the rays  $m$  of a sheaf, hence indirectly on the points of a plane. In my paper "Congruences of twisted curves in connection with a cubic transformation" (These Proceedings, Vol. 11, p. 84) I have shown how the properties of  $\Gamma$  appear through the transformation in question.

Any ray  $m$  of the sheaf about a point  $M$  is a bisecant of one  $k^3$  of  $\Gamma$ ; the point of intersection of  $m$  and an image plane may be considered as the image of  $k^3$ . I have treated this representation in a paper in Vol. 30, p. 850, of these Proceedings<sup>2)</sup>.

If we consider  $\Gamma$  as the product of two pencils of quadratic cones that have a base generatrix in common, there arises a representation of  $\Gamma$  on the points of a field if we bring both pencils into projective correspondence with two pencils of the image plane; in this case the point of intersection of two rays of these plane pencils may be considered as the image of the  $k^3$  that is produced by the corresponding cones<sup>3)</sup>.

In what follows we shall treat two other representations.

§ 2. The congruence  $\Gamma$  with base points  $B_k$  ( $k = 1, 2, 3, 4, 5$ ) contains ten systems of composite figures  $k^3$ . For the line  $B_1 B_2$  forms a  $k^3$  with any conic  $k^2$  in the plane  $\beta_{345}$  ( $B_3 B_4 B_5$ ) that passes through  $B_3, B_4, B_5$  and the point of intersection  $C_{12}$  of  $b_{12}$  ( $B_1 B_2$ ) with  $\beta_{345}$ ; let us indicate this system of curves  $k^3$  by  $\Sigma_{12}$ .

Let  $r$  be a straight line through  $B_5$ ,  $R$  its point of intersection with the plane  $\varrho$ . I shall consider  $R$  as the image of the  $k^3$  that touches  $r$  (at  $B_5$ ).

The transits  $S_1, S_2, S_3, S_4$  of the lines  $b_{15}, b_{25}, b_{35}, b_{45}$  are evidently singular image points; each of them represents a system  $\Sigma$ .

The other six systems  $\Sigma$  are represented on point ranges; the  $k^3$  of

<sup>1)</sup> We mean the transformation  $x_1 x'_1 = x_2 x'_2 = x_3 x'_3 = x_4 x'_4$ .

<sup>2)</sup> A Representation of the Congruence of REYE.

<sup>3)</sup> This representation has been investigated by H. P. HOESTRA ("Onderzoek van enige bilineaire congruenties van kubische ruimtekrommen door een afbeelding op het puntenveld", thesis for the doctorate, Utrecht 1928, p. 45).

$\Sigma_{12}$  have their images on the line  $s_{12}$  that joins  $S_3, S_4$ . The point of intersection of  $s_{12}$  and  $s_{34}$  is the image of the  $k^3$  that consists of  $b_{12}$ ,  $b_{34}$  and the line through  $B_5$  that rests on  $b_{12}$  and  $b_{34}$ .

§ 3. Let us now consider the system  $\Lambda$  of the  $k^3$  that cut the line  $l$ . As  $l$  rests on one  $k^2$  of any system  $\Sigma$ , the image curve  $\lambda$  of  $\Lambda$  has only the points  $S_3, S_4$  and the image  $R$  of a  $k^3$  belonging to  $\Sigma_{12}$  in common with  $s_{12}$ ; it is, therefore, a cubic.

Between the points of  $\lambda^3$  and the points of  $l$  there exists a correspondence (1,1); accordingly  $\lambda^3$  is *rational*. Its double point is the image of the  $k^3$  that has  $l$  as bisecant.

Two curves  $\lambda^3$  have 5 points  $R$  in common; consequently on two arbitrary lines there rest five  $k^3$  and the curves of  $\Lambda$  form a surface of the *fifth* degree.

The intersection of  $\Lambda^5$  and  $\beta_{123}$  consists of the lines  $b_{12}, b_{13}, b_{23}$  and a conic.

§ 4. Let us also consider the system  $\Phi$  of the  $k^3$  that touch a plane  $\varphi$ .  $\beta_{123}$  contains two conics  $k^2$  that touch  $\varphi$ ; the locus of the points where  $\varphi$  is touched by curves of  $\Gamma$  is, therefore, a conic  $\varphi^2$ .

As any system  $\Sigma$  contains two  $k^3$  of  $\Phi$ , the image curve of  $\Phi$  has double points in  $S_3$  and  $S_4$  and  $s_{12}$  contains two more points  $R$  of that curve. Because of the (1,1) correspondence between  $\varphi^6$  and  $\varphi^2$  the image curve of  $\Phi$  is a *rational*  $\varphi^6$ ; the six double points which it contains outside the points  $S$ , are images of the  $k^3$  that osculate  $\varphi$ .

Accordingly an arbitrary plane *osculates six* curves of  $\Gamma$ . Two curves  $\varphi^6$  have 20 points  $R$  in common; there are, therefore, *twenty*  $k^3$  that touch two planes.

A curve  $\lambda^3$  has ten points  $R$  in common with  $\varphi^6$ ; accordingly there are *ten*  $k^3$  that touch a plane and cut a line. And the curves of  $\Phi$  form a surface of the *tenth* degree.

A point range  $(R)$  is the image of the system of the  $k^3$  that touch a plane pencil  $(r)$ ; these  $k^3$  form a cubic surface, for  $(R)$  has three points in common with a  $\lambda^3$ .

§ 5. In order to arrive at another representation of  $\Gamma$  we pass a plane  $\varrho$  through  $b_{45}$  and we consider the third point of intersection  $R$  of a  $k^3$  with  $\varrho$  as the image of that  $k^3$ .

The transits  $S_{12}, S_{13}, S_{23}$  of  $b_{12}, b_{13}, b_{23}$  are singular points;  $S_{12}$  is e.g. the image of all  $k^3$  of the system  $\Sigma_{12}$ .

A  $k^3$  of  $\Sigma_{45}$  is represented by the points of  $b_{45}$  and by the point  $R$  which its  $k^2$  has besides in common with  $\varrho$ .

The image curve of the system  $\Lambda$  passes evidently through the points  $S$ ; with the line through these points it has besides only the point  $R$  in common that is the intersection of a  $k^2$  of  $\Sigma_{45}$  and  $l$ ; it is, therefore,

a *rational* curve  $\lambda^4$ . It has double points in  $B_4$  and  $B_5$ ; the third double point is the image of the  $k^3$  that has  $l$  as chord.

Two curves  $\lambda^4$  have five points  $R$  in common outside the five singular points; accordingly there are *five*  $k^3$  that rest on two lines.

§ 6. The singular point  $B_4$  is the image of all  $k^3$  that touch  $\varphi$  at  $B_4$ ; these curves evidently form a quadratic surface  $\beta^2_4$ . The  $k^3$  on  $\beta^2_4$  define an involution  $I^3$  on the conic in which a plane  $\varphi'$  cuts  $\beta^2_4$ . Accordingly on  $\beta^2_4$  there lie four  $k^3$  that touch  $\varphi$  and  $B_4$  is a quadruple point of the image curve of the system  $\Phi$ ; this is consequently a *rational* curve  $\varphi^8$  with quadruple points  $B_4$  and  $B_5$ . It has double points in the points  $S_{kl}$ , for the corresponding system  $\Sigma_{kl}$  contains two conics that touch  $\varphi$ . The six other double points of  $\varphi^8$  are the images of the *six*  $k^3$  that osculate  $\varphi$ .

**Mathematics. — Linear Congruences of Twisted Cubics that Cut at least one Fixed Line Twice.** By Prof. JAN DE VRIES.

(Communicated at the meeting of May 26, 1928).

§ 1. If the curves  $\varrho^3$  of a congruence  $\Gamma$  cut a fixed line  $b$  twice, and through an arbitrary point there passes only one  $\varrho^3$ ,  $\Gamma$  may be represented on the points of a plane  $\beta$  through  $b$ . As the image of  $\varrho^3$  we consider the point  $R$  outside  $b$  that it has in common with  $\beta$ .

If the curves of  $\Gamma$  pass through four fixed points  $A_k$ <sup>1)</sup> the curves that cut  $b$  in  $B$ , lie on the quadratic cone  $\beta^2$  that has the lines  $b$  and  $BA_k$  as generatrices (any  $\varrho^3$  has seven points in common with  $\beta^2$ ). This system is represented on a line  $r$  of  $\beta$ .

Let  $k$  be a chord of a  $\varrho^3$ . The hyperboloid  $H$  through three of the points  $A$  that contains  $b$  and  $k$ , passes through  $\varrho^3$ . It contains  $\infty^1$  curves  $\varrho^3$ ; for if we project  $H$  out of one of its points  $O$  on a plane, this contains a pencil of curves  $c^3$  all of which have a double point in one of the two cardinal points of the representation and pass through the second cardinal point and the images of the points  $A$ . These  $c^3$  are the images of curves  $\varrho^3$  through  $A$  that have the lines of the scroll to which  $b$  belongs, as bisecants. This pencil of  $\varrho^3$  is represented on the line  $r$  that  $H$  has still in common with  $\beta$ .

§ 2. To  $\Gamma$  belong four systems  $\Sigma_k$  of figures each of which consists of a line and a conic  $\varrho^2$ . The system  $\Sigma_1$  is formed by the conics in the plane  $a_{234}$  that have the points  $A_2, A_3, A_4$  and the transit of  $b$  as basis and each of which is completed by a ray  $t_1$  of the plane pencil about  $A_1$  in the plane  $A_1b$ . It is represented on a line  $c_1$ .

The four lines  $c_k$  form a quadrilateral; each of the six angular points is the image of a figure that belongs to two of the systems  $\Sigma$  and, accordingly, consists of three lines. Hence  $\Gamma$  contains six figures each of which consists of two crossing lines and one line that cuts them.

The system  $\Lambda$  of the  $\varrho^3$  that cut a line  $l$ , is represented on a conic  $\lambda^2$ . For the image line of a conic  $\beta^2$  contains the images  $R$  of the two  $\varrho^3$  belonging to  $\beta^2$  that rest on  $l$ . On two lines  $l_1$  and  $l_2$  there rest, therefore, four  $\varrho^3$  and the system  $\Gamma$  lies on a surface  $\Lambda^4$ . On this  $b$  is a double line. A  $\varrho^3$  outside  $\Lambda^4$  can only cut  $\Lambda^4$  on  $b$  and in  $A_k$ ; hence  $\Lambda^4$  has double points in  $A_k$ .

<sup>1)</sup> Another representation of the congruence  $(4A, b)$  has been treated by Dr. G. SCHAAKE in a paper in these Proceedings, Vol. 28, p. 776.

An arbitrary line  $l$  cannot be a chord of a  $\varrho^3$  belonging to  $\Gamma$ ; for the scrolls on the hyperboloids  $H$  together form a complex. The complex-cone projects a  $\varrho^3$ , is therefore quadratic, and the  $A_k$  are *cardinal points*; the complex is accordingly *tetrahedral*.

§ 3. The  $\varrho^3$  of an  $H$  define an  $I^3$  on the conic that  $H$  has in common with a plane  $\varphi$ ; four of these curves touch  $\varphi$ . Accordingly on the image line  $r$  there lie four points  $R$  originating from curves  $\varrho^3$  that belong to the system  $\Phi$  which is formed by the  $\varrho^3$  that touch a plane  $\varphi$ . Consequently the *image curve* of  $\Phi$  is a  $\varphi^4$ .

As  $\varphi^4$  has eight points  $R$  in common with a  $\lambda^2$ , the system  $\Phi$  lies on a surface  $\Phi^8$  on which  $b$  is *quadruple* and the  $A_k$  are *quadruple* points.

The points of contact of the  $\varrho^3$  lie on a curve  $\varphi^3$  that has a double point on  $b$ . For a plane  $\varphi$  through  $l$  has besides a  $\lambda^3$  in common with  $A^4$  and any point of intersection of  $\lambda^3$  and  $l$  is a point of contact of a  $\varrho^3$  of  $\Phi$ . Between the curves  $\varphi^3$  and  $\varphi^4$  there exists a (1,1)-correspondence; hence  $\varphi^4$  has three double points. Consequently  $\varphi$  is a *plane of osculation* for three curves of  $\Gamma$ .

§ 4. Let us now consider the congruence  $\Gamma$  of which the curves  $\varrho^3$  pass through the cardinal points  $A'$  and  $A''$  and have the lines  $b_1, b_2, b_3$  as cardinal chords (congruence of STUYVAERT).

It contains six systems  $\Sigma$  of composite figures. The line  $a''_{23}$  through  $A''$  that cuts  $b_2$  and  $b_3$ , forms figures  $\varrho^3$  with any conic  $\varrho^2$  in the plane  $A_1 b_1$  which passes through  $A_1$  and rests on  $b_2, b_3, a''_{23}$ . If the image plane  $\beta$  passes through  $b_1$ , the point  $S' \equiv a''_{23} \beta$  is the image of all  $\varrho^3$  of this system  $\Sigma'_1$ ; this point is, therefore, *singular* for the representation.

Analogously the *singular* point  $S'' \equiv a'_{23} \beta$  is the image of the system  $\Sigma''_1$  formed by  $a'_{23}$  and the conics in the plane  $A'' b_1$ .

The systems  $\Sigma'_2, \Sigma''_2, \Sigma'_3, \Sigma''_3$  are represented on lines  $c'_2, c''_2, c'_3, c''_3$ .

The  $\varrho^3$  through the point of intersection  $S_2$  of  $b_2$  and  $\beta$  lie on the hyperboloid  $H_2$  that contains  $b_1, b_3, S_2, A'$  and  $A''$ ; they have the *singular point*  $S_2$  as image.

Analogously  $S_3 \equiv b_3 \beta$  is the image of the  $\varrho^3$  on the hyperboloid  $H_3$  that contains  $b_1, b_2, S_3, A'$  and  $A''$ .

The system  $\Sigma'_2$  consists of the line  $a''_{13}$  and the conics in the plane  $A' b_2$  through  $A'$  that rest on  $b_1, b_3$  and  $a''_{13}$ . To  $(\varrho^2)$  belongs the pair of lines of which  $a'_{23}$  is one of the lines, which, accordingly, with  $a'_{13}$  forms a figure that belongs at the same time to  $\Sigma''_1$ . The *image line*  $c'_2$  contains, therefore, the point  $S''$ ; it contains at the same time the point  $S_2 \equiv b_2 \beta$ .

Analogously  $c''_2 \equiv S' S_2, c'_3 \equiv S'' S_3, c''_3 \equiv S' S_3$ .

§ 5. The conics through  $A'$  and  $A''$  that rest on  $b_1, b_2, b_3$ , lie on a *dimonoid*  $\Delta^4$  with double torsal line  $A'A''$ . They are completed to figures

of  $\Gamma$  by the transversals  $t$  of  $b_1, b_2, b_3$ . On  $\Delta^4$  lie six pairs of lines each of which has one of the lines  $a'_{kl}, a''_{kl}$  as component part. The *image curve*  $\delta^3$  of the system contains, therefore, the points  $S'$  and  $S''$ , and evidently also the points  $S_2$  and  $S_3$ .

The double point of  $\delta^3$  lies on  $A'A''$  and is the image of the  $\varrho^3$  that consists of  $A'A''$  and the two lines  $t$  that cut  $A'A''$ .

Each of the points  $S', S''$  is the image of three figures, each of which consists of three lines, each of the points  $S_2$  and  $S_3$  is the image of two such figures; finally also the points  $c'_3 c'_2$  and  $c'_2 c'_3$  are the images of figures consisting of three lines. Hence  $\Gamma$  contains thirteen  $\varrho^3$  formed by three lines.

§ 6. A line  $l$  cuts two  $\varrho^3$  of  $H_2$  and two of  $H_3$ , hence  $S_2$  and  $S_3$  are double points of the image curve  $\lambda$  of the system  $\Lambda$ . This evidently contains the points  $S'$  and  $S''$ , for  $l$  cuts one  $\varrho^2$  of each of the systems  $\Sigma$ . Accordingly the curve  $\lambda$  has the double point  $S_2$ , the point  $S''$  and a point  $R$  in common with the image line  $c'_2$ ; it is, therefore, a  $\lambda^4(S_3^2 S_2^2 S' S'')$ . Being a rational curve it has a *third* double point; this is the image of a  $\varrho^3$  that cuts  $l$  twice. Any line is, therefore, a *bisecant* of one curve  $\varrho^3$ .

Two curves  $\lambda^4$  have six points  $R$  in common; accordingly on two lines there rest six  $\varrho^3$  and the curves  $\varrho^3$  that are cut by  $l$ , form a surface  $\Lambda^6$ . On this  $b_2$  and  $b_3$ , but then also  $b_1$ , are *double lines* and  $A', A''$  are *triple points*.

Two surfaces  $\Lambda^6$  have the double lines  $b$ , the six lines  $a'_{kl}, a''_{kl}$  and six curves  $\varrho^3$  in common. The curves  $\lambda^4(S_2^2 S_3^2 S' S'')$  and  $\delta^3(S_2 S_3 S' S'')$  have six points  $R$  in common; these are the images of six composite  $\varrho^3$ ; of four of these  $\varrho^3$  the conic rests on  $l$ , of the other two the line  $t$ .

§ 7. A hyperboloid  $H$  contains again four  $\varrho^3$  that touch a plane  $\varphi$ , a system  $\Sigma$  contains two of them. The *image curve* of the system  $\Phi$  is, therefore, a  $\varphi^8(S_2^4 S_3^4 S'^2 S''^2)$ . It has 12 points  $R$  in common with a  $\lambda^4$ ; accordingly the system lies on a surface  $\Phi^{12}$  on which the lines  $b$  are *quadruple*, the lines  $a'_{kl}, a''_{kl}$  *double*.

A plane  $\varphi$  through  $l$  has still a curve  $\lambda^5$  in common with  $\Lambda^6$ ; this cuts  $l$  in the two points of support of the  $\varrho^3$  for which  $l$  is a chord and in the points of contact of three curves  $\varrho^3$  with  $\varphi$ . Hence the locus of the points of contact is a curve  $\varphi^3$  of the genus 1. Accordingly also  $\varphi^8$  has the genus 1; it has, therefore, six double points besides the point  $S$ . Consequently a plane  $\varphi$  is *osculated* by six curves  $\varrho^3$ .

§ 8. Let us now consider the  $\Gamma$  of which the  $\varrho^3$  pass through the point  $A$  and have four fixed bisecants  $b_k$  (congruence of GODEAUX).

Let the image plane  $\beta$  again pass through  $b_1$ .  $\Gamma$  contains the system  $\Sigma_1$  formed by conics  $\varrho^2$  in the plane  $Ab_1$  and transversals  $t_1$  of  $b_2, b_3, b_4$ ; it is represented on the conic  $\gamma_1^2$  in which the hyperboloid  $H_1$  (through  $b_2, b_3, b_4$ ) cuts  $\beta$ .

The analogous system  $\Sigma_2$  is represented on a line  $c_2$ , the systems  $\Sigma_3$  and  $\Sigma_4$ , on lines  $c_3$  and  $c_4$ .

The curves through  $A$  and  $S_2 \equiv b_2\beta$  that have  $b_1, b_3, b_4$  as chords and rest on  $l$ , form (§ 6) a surface  $O^6$  on which  $S_2$  is a triple point. It contains therefore three  $\varrho^3$  that cut  $b_2$  once more and accordingly belong to  $\Gamma$ . Consequently the  $\varrho^3$  of  $\Gamma$  through  $S_2$  form a cubic surface  $O_2^3$ . Analogously the singular points  $S_3 \equiv b_3\beta$  and  $S_4 \equiv b_4\beta$  are images of systems that lie on surfaces  $O_3^3$  and  $O_4^3$ .

The image lines  $c_2, c_3, c_4$  evidently contain resp. the singular points  $S_2, S_3, S_4$ .

Three of the  $\varrho^3$  that cut  $l$  belong to  $\Sigma_2$  for  $l$  rests on one  $\varrho^2$  and on two lines  $t_2$ . The image curve  $\lambda$  has three points  $R$  and the triple point  $S_2$  in common with  $c_2$ ; it is, therefore, a  $\lambda^6(S_2^3 S_3^3 S_4^3)$ . It contains also a double point, image of a  $\varrho^3$  that cuts  $l$  twice.

On two lines  $l$  there rest nine  $\varrho^3$  and the system  $\Lambda$  lies on a  $\Lambda^9$  with triple lines  $b_1, b_2, b_3, b_4$  and triple point  $A$ .

§ 9. Let  $\Delta_1$  be the surface formed by the  $\varrho^2$  of  $\Gamma$  that rest on the four lines  $b$  and on their transversal  $t'$ ; analogously  $\Delta_2$  the surface corresponding to the second transversal  $t''$ . Evidently the systems  $\Delta_1$  and  $\Delta_2$  have only one  $\varrho^3$  in common, the figure consisting of  $t', t''$  and their transversal through  $A$ . Accordingly their image curves have one point  $R$  in common; as they contain the points  $S_2, S_3, S_4$  they are conics  $\delta_1^2$  and  $\delta_2^2$ .

$\Delta_1$  contains a  $\varrho^2$  that passes through the point  $b_2 t'$ ; the  $\varrho^3$  which it forms with  $t'$ , belongs at the same time to  $\Sigma_2$ ; its image point  $R$  lies, therefore, also on  $c_2$ . In fact this has only one point  $R$  in common with  $\delta_1^2$  outside  $S_2$ .

The curves  $\delta_1^2(3S)$  and  $\lambda^6(3S^3)$  have three points  $R$  in common; hence  $\Delta_2$  is a cubic scroll with directrices  $t', t''$  ( $\Delta_1^3$  has one more line in common with the plane of a  $\varrho^2$ ).

There are seven  $\varrho^3$  composed of three lines; one of them consists of  $t', t''$  and  $a$ ; the other six are represented in the angular points of the quadrilateral formed by the lines  $c$ .

§ 10. The system  $\Sigma_2$  contains two  $\varrho^2$  that touch a given plane  $\varphi$  and two figures  $\varrho^3$  of which the point  $\varrho^2 t_2$  lies in  $\varphi$ ; each of these must be counted twice.

The surface  $O_2^3$  has a curve  $\varphi^3$  of genus one in common with  $\varphi$ ; the  $I^3$  which the curves  $\varrho^3$  define on this, has six coincidences;  $S_2$  is, therefore, sextuple on the image curve of the system  $\Phi$ . Of this curve  $c_2$  contains the 6-fold point  $S_2$  and 6 points  $R$ ; accordingly it is a  $\varphi^{12}(3S^6)$ . It has 18 points  $R$  in common with a  $\lambda^6(3S^3)$ ; the system  $\Phi$  lies, therefore, on a  $\Phi^{18}$  with sextuple lines  $b$  and sextuple point  $A$ .

§ 11. In the representation of the congruence  $\Gamma$  that has five cardinal

chords  $b$ , the points  $S_2, S_3, S_4$  and  $S_5$  in which  $b_2, b_3, b_4, b_5$  cut  $\beta$ , are singular<sup>1)</sup>.

The  $\varrho^3$  through  $S_2$  that have  $b_1, b_3, b_4$  and  $b_5$  as chords and rest on  $l$ , form (§ 8) an  $O^9$  on which  $S_2$  is a triple point. Accordingly there are six  $\varrho^3$  of this system that cut  $b_2$  once more. Hence the  $\varrho^3$  of  $\Gamma$  that are represented in  $S_2$ , form a surface  $\beta_2^6$ .

The hyperboloid  $H_{234}$  with directrices  $b_2, b_3, b_4$  contains  $\infty^1$  curves of  $\Gamma$ ; they all pass through the four points in which  $H_{234}$  is cut by  $b_1$  and  $b_5$ . The image curve of this system is a conic  $\gamma^2_{234}$  through the singular points  $S_2, S_3$  and  $S_4$ .

The analogous hyperboloid  $H_{145}$  is represented on a line  $c_{145}$ . The hyperboloids  $H_{145}$  and  $H_{234}$  have a  $\varrho^3$  besides  $b_4$  in common that has the second point of intersection of  $c_{145}$  and  $\gamma^2_{234}$  as image. The curves  $\gamma^2_{234}$  and  $\gamma^2_{235}$  have two points  $R$  in common; these are the images of two figures that consist of a  $\varrho^2$  and one of the transversals  $t', t''$  of  $b_2, b_3, b_4, b_5$  and are, accordingly, singular points for the representation; we shall indicate them by  $S'$  and  $S''$ . That they are singular appears thus: any  $\varrho^2$  that has  $b_1$  as chord and rests on  $b_2, b_3, b_4, b_5$  and  $t'$ , forms with  $t'$  a  $\varrho^3$  belonging to  $\Gamma$ ; all these figures are represented in the point of intersection  $S'$  of  $t'$  and  $\beta$ . They form a surface  $O'_1^4$  with double line  $b_1$ .<sup>2)</sup>

The analogous system  $O'_2^4$  with double line  $b_2$  is represented on the curve  $\omega_2^3(S_2^2)$  which  $\beta$  has in common with this surface.  $O'_2^4$  contains the lines  $t'$  and  $t''$ ; for the plane  $b_2 t'$  contains one line that cuts  $b_1$  and the transversal  $t'_{1345}$  and with this and with  $t'$  forms a  $\varrho^3$ .

Accordingly the image curve of this system is an  $\omega_2^3(S_2^2 S_3 S_4 S_5 S' S'')$ .

§ 12. The system  $A$  has an image curve of which  $S'$  and  $S''$  are quadruple points, the points  $S_2, S_3, S_4, S_5$  are sextuple points. Besides  $S', S'', S_2, S_3, S_4$   $\gamma^2_{234}$  contains the images  $R$  of the two  $\varrho^3$  of  $H_{234}$  that rest on  $l$ ; accordingly it has 28 points in common with the image curve  $\lambda$  and consequently it is a  $\lambda^{14}(S_2^6 S_3^6 S_4^6 S_5^6 S'^4 S''^4)$ .

The line  $S_4 S_5$ , image of  $H_{145}$ , contains two points  $R$ ; also this shows that the order of  $\lambda$  is 14.

Two curves  $\lambda$  have 20 points  $R$  in common; accordingly on two lines  $l$  there rest 20 curves  $\varrho^3$ .

As  $\lambda^{14}$  is rational it has six double points outside the points  $S$ ; there are, therefore, six curves  $\varrho^3$  that have six given lines as chords.

The system  $A$  lies on a surface  $A^{20}$  on which the five lines  $b$  are sextuple and the ten transversals  $t$  are quadruple.

<sup>1)</sup> I have treated another representation of this congruence in my paper: "The Congruence of the twisted Cubics that cut five given lines twice". (These Proceedings, Vol. 31, p. 454).

<sup>2)</sup> L.c. p. 454.

§ 13. Let us also consider the congruence  $\Gamma$  with cardinal chord  $b$  and the cardinal points  $A_1, A_2, A_3$  of which the curves  $\varrho^3$  rest on the line  $c_1$  that passes through  $A_1$  and the line  $c_2$  that passes through  $A_2$ <sup>1)</sup>.

The curves through the point  $S_1(c_2\beta)$  and the points  $A$  that cut  $b$  twice and rest on  $l$ , form an  $O^4$  with double point  $A_2$  (§ 2); it contains two  $\varrho^3$  that cut  $c_2$  once more. Accordingly the curves of  $\Gamma$  that pass through  $S_1$ , form a quadratic surface  $O_1^2$ . Analogously there is an  $O_2^2$  of which the curves  $\varrho^3$  are represented in the singular point  $S_2(c_2\beta)$ .

In the plane  $A_3b$  there lies one  $(\varrho^2)$  of which any individual is completed to a  $\varrho^3$  by the line  $a'_1t_2(A_1A_2)$ . All these  $\varrho^3$  have the singular point  $S(a_{12}\beta)$  as image.

To this system  $\Sigma_3$  there belong three figures  $\varrho^3$  that consist of three lines. In the first place the system of  $a_{12}$ , the transversal through  $A_3$  of  $a_{12}$  and  $b$  and a line in  $A_3b$  that cuts  $c_1$  and  $c_2$ . This figure may also be considered as the system of a pair of lines in the plane  $a_{123}(A_1A_2A_3)$  and a transversal of  $b, c_1, c_2$ . It belongs, therefore, at the same time to the system  $\Sigma_{123}$  of the  $\varrho^3$  that consist of a  $\varrho^2$  in  $a_{123}$  and a transversal of  $b, c_1, c_2$ . Accordingly the image of  $\Sigma_{123}$  is a line  $d_{123}$  through  $S$ .

$\Sigma_3$  contains also the  $\varrho^3$  that consists of  $a_{12}, A_3C_1$  and  $A_{12}C_2(C_1, C_2$  and  $A_{12}$  are base points of the pencil  $(\varrho^2)$ ). As  $a_{12}$  forms a pair of lines in the plane  $A_1c_2$  with  $A_{12}C_2$ , this  $\varrho^3$  belongs at the same time to the system  $\Sigma_2$  of which the  $\varrho^3$  consist of a  $\varrho^2$  in  $A_1c_2$  and the line  $a'_3$  through  $A_3$  cutting  $c_1$  and  $b$ .

The system  $\Sigma_2$  is represented on the line  $d_2$  that joins the points  $S$  and  $S_2(c_2\beta)$ .

Analogously the line  $d_1(SS_1)$  is the image of the system  $\Sigma_1$ . The conics  $\varrho^2$  in  $A_3c_1$  through  $A_1$  and  $A_3$  that cut  $b$  and  $c_2$  and are associated to a line  $t_2$  through  $A_2$  that rests on  $b$ , form a system  $\Sigma'$  that has a line  $d'$  through  $S_1$  as image.

Analogously there is a line  $d''$  through  $S_2$  that represents the system  $\Sigma''$ .

The lines  $b, c_2, a'_2$  (line through  $A_2$  cutting  $b$  and  $c_1$ ),  $A_1$  and  $A_3$  define a hyperboloid  $H_2$ ; it contains the system  $\Sigma_{13}$  of the figures that consist of  $a'_2$  and a conic through  $A_1$  and  $A_3$ . This system is represented on a line  $d_{13}$  through the point  $S_2(c_2\beta)$ .

Analogously there is a system  $\Sigma_{23}$  of figures that consist of the line  $a''_1$  (through  $A_1$  cutting  $b$  and  $c_2$ ) and a conic through  $A_2$  and  $A_3$ , and that lie on the hyperboloid  $H_1$  that is defined by  $b, c_1, a''_1, A_2$  and  $A_3$ . The image line  $d_{23}$  contains the point  $S_1(c_1\beta)$ .

§ 14. The image curve  $\lambda$  of the system  $A$  of the  $\varrho^3$  that rest on  $l$ , passes through  $S$  and has double points in  $S_1$  and  $S_2$ . The system  $\Sigma_1$

<sup>1)</sup> That this congruence is linear appears thus. Let  $P$  be an arbitrary point; the lines  $A_1A_2, A_1A_3, A_1P$  and  $c_1$  form the basis of a pencil of quadratic cones. Analogously  $A_2A_1, A_2A_3, A_2P$  and  $c_2$  define a similar pencil. The two pencils define an  $I^2$  on  $b$  each; there is, therefore, one  $\varrho^3$  of  $\Gamma$  that has  $b$  as chord.

contains one  $\varrho^3$  that cuts  $l$ ; hence the image line  $d_1$  has a point  $R$ , the point  $S$  and twice the point  $S_1$  in common with  $\lambda$ , which is, accordingly, a  $\lambda^4(S S_1^2 S_2^2)$ . Being a rational curve it has still a *double point*, the image of the  $\varrho^3$  that has  $l$  as chord.

Two curves  $\lambda^4$  have *seven* points  $R$  in common; accordingly the system  $\Lambda$  lies on a surface  $\Lambda^7$ .

The intersection of  $\Lambda^7$  with  $a_{123}$  consists of three  $\varrho^2$  and the line  $a_{12}$ ; for  $l$  rests on three figures of  $\Sigma_{123}$  and on one  $\varrho^2$  of  $\Sigma_3$ . Hence  $A_3$  is a *triple point* and  $A_1$  and  $A_2$  are *quadruple points* on  $\Lambda^7$ ,  $c_1$  and  $c_2$  are *double lines* and  $b$  is a *triple line* (points of intersection of  $\Lambda^7$  and an arbitrary  $\varrho^3$ ). Further  $a'_2$  and  $a''_1$  are *double lines* and  $\Lambda^7$  contains the lines  $a_{12}$ ,  $a'_3$  and  $a''_3$  and 14 conics, 3 rays  $t_{12}$ , 2 rays  $t_1$  and 2 rays  $t_2$ .

The image curve of the system  $\Phi$  has  $S_1$  as quadruple point (system  $O_1^2$ ),  $S$  as double point (system  $\Sigma_1$ ); the image line  $d_1(S_1 S)$  of  $\Sigma_1$  contains two more points  $R$ ; hence the image curve of  $\Phi$  is a  $\varphi^8(S_1^4 S_2^4 S^2)$ . In connection with  $\lambda^4(S_1^2 S_2^2 S)$  we find that the curves of  $\Phi$  lie on a surface  $\Phi^{14}$ .<sup>1)</sup>

§ 15. Let us also consider the congruence of the  $\varrho^3$  that has  $b_1$  and  $b_2$  as cardinal chords,  $A_1$  and  $A_2$  as cardinal points, and where each of the  $\varrho^3$  cuts the line  $c_1$  through  $A_1$  and the line  $c_2$  through  $A_2$  once more.

The image plane  $\beta$  is again passed through  $b_1$ . The point  $S(b_2 \beta)$  is *singular*. The  $\varrho^3$  through  $A_1$ ,  $A_2$ ,  $S$  that cut  $c_1$ ,  $c_2$  once more, have  $b_1$  as chord and rest on  $l$ , form a surface  $\Lambda^7$  with triple point  $S$  (§ 14). Accordingly this contains *four*  $\varrho^3$  that cut  $b_2$  once more, and the curves represented in  $S$  form a surface  $O^4$ .

Also  $S_1(c_1 \beta)$  is *singular* and is the image of the  $\varrho^3$  that lie on a hyperboloid  $H_1$  which is defined by  $A_1$ ,  $A_2$ ,  $S_1$ ,  $b_1$  and  $b_2$ .

Analogously  $S_2(c_2 \beta)$  is *singular* and is the image of a system  $\varrho^3$  on the hyperboloid  $H_2$  through  $A_1$ ,  $A_2$ ,  $S_2$ ,  $b_1$ ,  $b_2$ .

The plane  $A_1 b_1$  contains a  $(\varrho^2)$  of which the  $\varrho^2$  pass through  $A_1$  and rest on  $b_2$ ,  $c_2$  and on the transversal  $a'_{22}$  through  $A_2$  of  $b_2$  and  $c_1$ . Each of them forms with  $a'_{22}$  a  $\varrho^3$  of  $\Gamma$  (system  $\Sigma_{22}$ ) and is represented in the *singular* point  $S_{22}(a'_{22} \beta)$ .

Analogously the *singular* point  $S_{12}$  lying on the transversal  $a''_{12}$  through  $A_1$  of  $b_2$  and  $c_2$  is the image of the system  $\Sigma_{12}$  of which the conics  $\varrho^2$  lie in the plane  $A_2 b_1$ .

The system  $\Sigma'_{21}$  formed by the line  $a'_{21}$  (through  $A_2$  and cutting  $b_1$ ,  $c_1$ ) with a  $(\varrho^2)$  in the plane  $A_1 b_2$  is represented on the points of the line  $d'_{21}$  that  $A_1 b_2$  has in common with  $\beta$ . This line contains the points  $S(b_2 \beta)$  and  $S_{12}(a''_{12} \beta)$ ; to  $\Sigma'_{21}$  there also belongs a figure that contains  $a'_{21}$  and  $a''_{12}$ .

<sup>1)</sup> The congruence can also be represented on the field of points of the plane  $A_1 c_1$ .

Analogously  $\Sigma''_{11}$  formed by  $a''_{11}$  (through  $A_1$  cutting  $b_1, c_1$ ) with a  $\varrho^2$  in the plane  $A_2 b_2$ , has an *image line*  $d''_{11}$  that passes through  $S_{22}$  and  $S$ .

§ 16. Any line  $t_1$  that rests on  $b_1, b_2$  and  $c_1$ , forms a  $\varrho^3$  with a  $\varrho^2$  in  $A_1 c_2$  that passes through  $A_1, A_2$  and cuts  $b_1, b_2, t_1$  (system  $\Sigma_1$ ). The *image line*  $d_1$  passes through  $S_2 (c_2 \beta)$ .

Analogously any transversal  $t_2$  of  $b_1, b_2, c_2$  forms a  $\varrho^3$  with a  $\varrho^2$  in  $A_2, c_1$  through  $A_1, A_2$  that cuts  $b_1, b_2, t_2$ . This system  $\Sigma_2$  has as image a *line*  $d_2$  through  $S_1 (c_1, \beta)$ .

Let  $t$  be one of the transversals of  $b_1, b_2, c_1, c_2$ . Any  $\varrho^2$  through  $A_1, A_2$  that cuts  $b_1, b_2$  and  $t$ , forms a  $\varrho^2$  of  $\Gamma$  with  $t$ . The  $\varrho^2$  lie on the *hyperboloid*  $H$  that is defined by  $A_1, A_2, b_1, b_2$  and a point of  $t$ . This system  $\Sigma$  is represented on the points of the *line*  $d$  which  $H$  has besides in common with  $\beta$ ;  $d$  contains the point  $S$ .

Analogously the second transversal,  $t^*$ , of  $b_1, b_2, c_1, c_2$  defines a system  $\Sigma^*$  with *image line*  $d^*$  that passes through  $S$ .

The line  $a_1$  through  $A_1$  cutting  $b_1, b_2$  forms a  $\varrho^3$  with any  $\varrho^2$  that has  $c_2$  as chord and rests on  $c_1, b_1, b_2, a_1$ . The conics through  $A_2$  and a point of  $c_2$  that cut  $b_1, b_2$  and  $a_1$ , form a surface  $O^2$ ; through the second point of intersection of  $O^2$  and  $c_1$  passes one of these  $\varrho^2$ . Hence  $c_2$  is a single line on the locus of the  $\varrho^2$  that are completed by  $a_1$  to curves of  $\Gamma$ , and this is a cubic *monoid*  $O_1^3$  with double point  $A^2$ . This system  $\Sigma'$ , is represented on a *conic*  $\delta_1^2$  that passes through the points  $S, S_1, S_2$  and through  $S_{22}$  ( $a'_{22}$  lies on  $O_1^3$ ).

Analogously the system  $\Sigma''$ , lying on the *monoid*  $O_2^3$ , is represented on a conic  $\delta_2^2$  that contains the points  $S, S_1, S_2$  and  $S_{12}$ .

§ 17. The system  $\Sigma_1$  contains a figure consisting of  $a''_{12}$ , a line  $t_1$  and the transversal through  $A_2$  of  $b_1$  and  $t_1$ . The transversal forms a pair of lines of the plane  $A_2 b_1$  with  $t_1$ ; accordingly the  $\varrho^3$  belongs at the same time to the system  $\Sigma_{12}$ . Hence the *image line*  $d_1$  of  $\Sigma_1$  joins the points  $S_2$  and  $S_{12}$ .

$\Sigma_1$  contains three  $\varrho^3$  of the system  $\Lambda$ . As  $l$  cuts two  $\varrho^3$  of the system  $H_2$  and one  $\varrho^3$  of  $\Sigma_{12}$ , the image curve of  $\Lambda$  is a  $\lambda^6$ . On  $l$  there rest four  $\varrho^3$  of the system  $O^4$ ; hence  $\Lambda$  has a  $\lambda^6 (S^4 S_1^2 S_2^2 S_{12} S_{22})$  as *image*.

Two curves  $\lambda^6$  have ten points  $R$  in common; consequently the curves that rest on  $l$ , form a  $\Lambda^{10}$ . On this  $b_1, b_2$  are *quadruple*,  $c_1, c_2, t, t^*$  *double*,  $a_1, a_2$  *triple*; besides  $\Lambda^{10}$  contains the lines  $a'_{22}, a''_{12}, a'_{21}, a''_{11}$ , three lines  $t_1$ , three lines  $t_2$  and 20 conics. Finally by noticing the points of intersection with an arbitrary  $\varrho^3$  of  $\Gamma$ , we find that  $A_1$  and  $A_2$  are *quintuple* points.

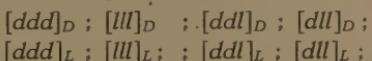
Two surfaces  $\Lambda^{10}$  have ten curves  $\varrho^3$  in common besides the lines  $b, c, a, a', a'', t, t^*$ .

**Chemistry. — On the spatial Arrangement and Rotatory Dispersion of optically active Complex Salts of Cobalt and Rhodium.** By Prof. Dr. F. M. JAEGER and Dr. H. B. BLUMENDAL.

(Communicated at the meeting of May 26, 1928).

§ 1. In the following pages some investigations are shortly reviewed, which were recently made in this laboratory with the purpose to answer the question: what changes in the rotatory dispersion of optically active complex salts of the *Triethylenediamine-Cobaltic*-type will be caused by gradually substituting the three molecules *ethylenediamine* in the complex ion by one, two or three molecules of another optically-active bivalent base, possessing itself, like *ethylenediamine*, a certain degree of axial symmetry? And furthermore: what will be the number of isomerides in these cases, and what will be their spatial arrangement, in connection with that of the active molecules introduced?

It may be remarked that already in the simplest case, i.e. that, in which all three *ethylenediamine*-molecules of the complex *Co*-, or *Rh*-cations are replaced by three molecules of an optically-active base, — the reaction may lead to a rather considerable number of isomers, if the *racemic* base, acting as *d,l*-base, be used in it. For, if the dextro- and levo-rotatory components be indicated by *d* and *l*, the racemate by *r*(= *d,l*), but the dextro-, respectively levorotatory configurations of the complex ion by *D* and *L*, — then the action of 3 molecules of the *racemic* base, being such that only an *odd* number of optically-active molecules can be used for the formation of each cation, can lead already, theoretically, to the following *eight* combinations:

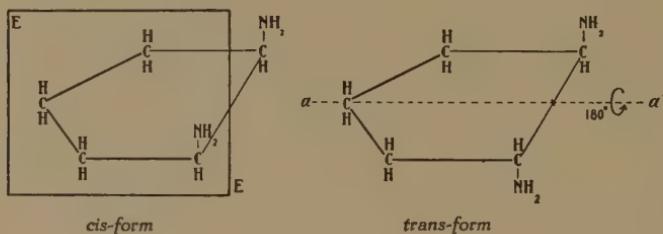


if no special causes be indicated beforehand, why one of these configurations should be impossible. As, moreover, these 8 cases represent in pairs each other's mirror-images, e.g.:  $[ddl]_D$  and  $[ddl]_L$ ; etc., — the said derivatives may be in pairs combined in aequimolecular quantities, yielding in this way 4 *racemates* and 24 *partial racemates*.

It is worth while remarking here, that all such cations, in which 3 *identical* molecules are present, — as is the case also in the *triethylenediamine*-complexes mentioned already, — will possess the rather high degree of symmetry  $D_3$ , this being the symmetry of *d*- and *l-quarz*, if only the substituents themselves have a single binary axis of symmetry. But in other cases, as e.g.:  $[ddl]$ ; etc., this degree of symmetry of the cation must be much less, even if the condition just mentioned be

fulfilled. In this case the complex ions will only have a single polar binary axis ( $C_2$ ). In both cases, however, these complex ions may be separated into two different mirror-images, because their symmetry is always a purely axial one.

§ 2. We have chosen as a bivalent base apparently well suited to the said purpose, the 1-2-Diamino-cyclopentane (=Cyclopentylene-diamine), — a substance unknown up till now. As may be seen from the following formulae, this compound may occur in a *cis*- and a *trans*-configuration:



of which, however, the *cis*-form, having a plane of symmetry perpendicular to the plane of the carbon-cycle, is not different from its mirror-image; while the *trans*-form, which possesses only a single binary axis of symmetry, situated in the plane of the carbon-cycle, — may under favourable circumstances, evidently be separated into non-surperposable mirror-images.

It appeared, however, pretty soon during the preparation of the new base, in which, in principal features, the following way was followed: *Trimethylenedicyanide* → *Glutaric Acid* → *Diethyl-Glutaric Ester* → *Diethyl-Cyclopentanedione-dicarbonic Ester* (obtained by condensation with *oxalic ester*) → *1-2-Cyclopentanedione* → *1-2-Cyclopentanedioxime* → (by reduction with an excess of *sodium* and *alcohol*) → *1-2-Diaminocyclopentane*, — that exclusively the *trans*-compound was formed. Because a reduction of the *dioxime* appeared to be possible only in an alkaline medium, — it being highly sensitive to acids, — no trace of the *cis*-derivative could hitherto be obtained, not even by catalytic reduction. The *trans*-configuration of the base obtained, — which boils under atmospheric pressure at  $170^\circ$  C. and which yields a beautifully crystallized *diacetyl*-derivative (mpt.:  $219^\circ$  C.), — was proved exactly by the possibility of separating it into two optical antipodes by means of *d*- and *l*-*tartaric acids*.

If, for instance, the racemic base be converted into its *bi-d-tartrate*, the *bitartrate* (mpt.  $143^\circ$ — $144^\circ$  C.) of the *levorotatory* base first crystallizes; etc. The boilingpoint (1 atm.) of this base is  $166^\circ$  C. The specific rotations of the pure optically-active base are: for red: —  $38^\circ$ , for violet: —  $113^\circ$ , 5.

We have also made a number of trials to convert the *trans*-base into

an *oxaryl*-derivative, either by means of *diethyloxalate*, or by means of *oxarylchloride*, and to separate from it the *cis*-derivative. But no positive results were obtained in this way: the *trans*-configuration seems to be so extremely stable in comparison with the *cis*-form, that this last appears to be re-arranged immediately into the first one. For this reason we will speak in the following page only of the *d*-, the *l*-, and the *rac*. *trans-1-2-Diaminocyclopentane*.

§ 3. Circumstances appear to be simplest, if we draw our attention in the first place to the complex *Rhodium*-salts, which are obtained in the reaction between *racemic trans-1-2-Diaminocyclopentane* and crystalline *sodium-rhodiumhexachloride* (hydrated, powdered). Subsequently the same reaction can be executed with each of both antipodes of the base; the products thus obtained serving to be immediately compared with the former ones.

In treating the solid  $Na_3RhoCl_6 \cdot 9H_2O$  with the *racemic* base, besides a small quantity of an orange-coloured by-product, soluble in alcohol<sup>1)</sup>, only a single white, crystalline product was obtained, which, by means of  $NaJ$ , could readily be precipitated as a colourless *iodide*, from which a whole series of other salts could be prepared by means of *silversalts* such as  $AgCl$ ,  $AgBr$ , etc. The *iodide* had the composition:  $\{Rho(Cptdine)_3\}I_3 + 1H_2O$ ; the corresponding *chloride* crystallizes with  $2H_2O$ .

By converting this *chloride* by means of *Ag-d-tartrate*, respectively *Ag-l-tartrate*, into the corresponding *chloro-d*-, resp. *chloro-l-tartrates*, it could be proved that the original salt is a true *racemic* compound, from which two optically-active antipodes could be separated. If *Ag-d-tartrate* was used, the less soluble fraction of the *chloro-d-tartrates* thus obtained, gave a *chloride*, which in solution appeared to be strongly *levorotatory*. By careful fractional crystallisation and by determining the complete curve of rotatory dispersion of these fractions, it could be proved with certainty, that no other components than the *dextro*-, respectively *levorotatory* salts mentioned, were present in the original reaction-product.

If now only the *d*-, or the *l*-base were used in the same reaction, then in each case only a *single* optically-active product appeared to be generated. By accurate measurement of the rotatory dispersion of both these salts and by comparing them with the products obtained by the fission of the *racemic* compound mentioned above, it could be proved

<sup>1)</sup> The orange-coloured by-product, treated with  $NaJ$ , gave an intensely orange-coloured *iodide*, which, on heating its alcoholic solution, was gradually decomposed under precipitation of the ordinary white *iodide*, mentioned above. By analysis we were able to prove that most probably a double compound is present here, consisting of 2 mol. of the white *iodide* and 1 mol. of a non-ionogenic complex, containing *Rho*, *Cl*, *alcohol* and *Diaminocyclopentane*.

with complete certainty, that in the *levorotatory* complex cation only three molecules of the *d*-base, in the *dextrorotatory* complex cation only three molecules of the *l*-base were present. No other salts than those of the ions:  $\{Rho(III)\}_D^{+}$  and  $\{Rho(ddd)\}_L^{+}$  were generated in the reaction between the racemic *Diaminocyclopentane* and  $Na_3RhoCl_6$ , 9 aq.

The complete curve of the rotatory dispersion of these salts is represented in Fig. 1.

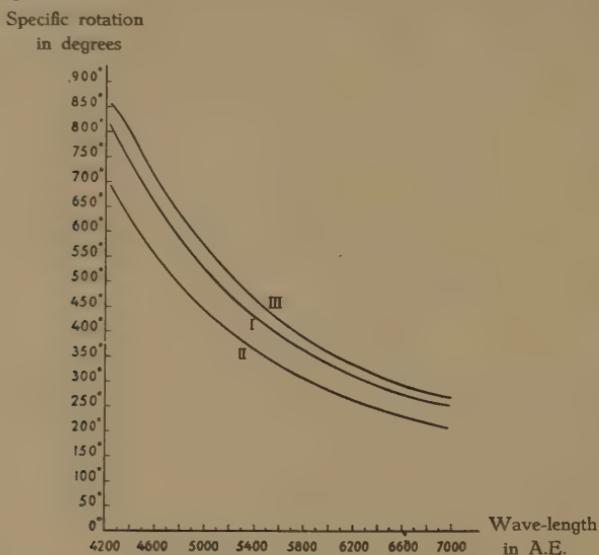
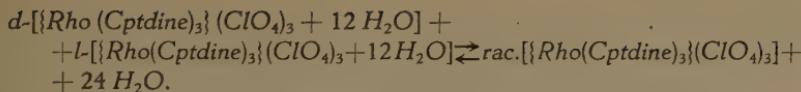


Fig. 1.  
Rotatory dispersion of  $\{Rho(l-Cptdine)_3\}Cl_3$  and its *d*- and *l*-Chloro-tartrates

From this it must be concluded, that the *levorotatory* configuration of the complex ion is incompatible with the presence of 3 *levorotatory* molecules of the base in it; and that, in the same way the presence of 3 *dextrorotatory* molecules of the base is incompatible with the *dextrorotatory* configuration of the complex *Rho*-ion.

In the case of the racemic *perchlorate*, which forms anhydrous crystals, a fission into the antipodes was observed by *spontaneous crystallisation* of the solutions at roomtemperature. The crystals deposited were octahedral, and, if selected and individually solved, they gave solutions, which were dextro-, or levorotatory. The determination of the solubilities of the racemic compound and of the antipodes taught us, that a case of spontaneous crystallisation was present here with a *minimum* transformation-temperature of  $48^{\circ} C.$  for the racemoid, the racemic compound becoming the stabler phase above this temperature (Fig. 2). The said transformation may be represented by:



The fission of the anhydrous racemic compound, in presence of the solution, into the highly hydrated antipodes must, therefore, evidently occur under an evolution of heat.

Solubility in grammes  
anhydrous salt in 100 cm<sup>3</sup> sol.

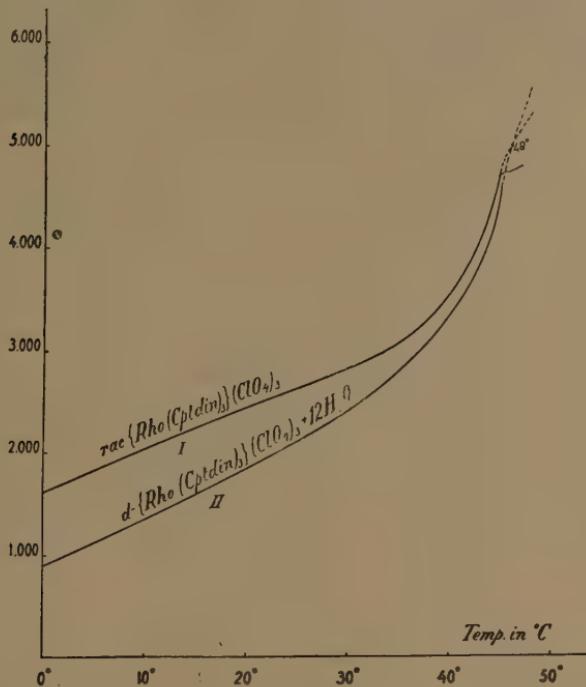


Fig. 2.

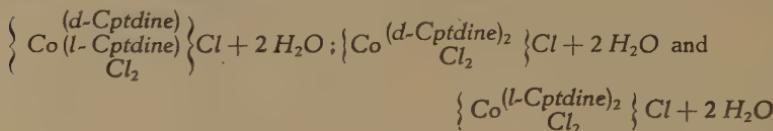
Solubility of rac. and dextro-rot.  $\{Rho(Cptdine)_3\}(ClO_4)_3$  at different temperatures.

The optically-active chlorides (+ 4H<sub>2</sub>O), nitrates (+ 4H<sub>2</sub>O), chlorates (+ 12H<sub>2</sub>O) and perchlorates (+ 12H<sub>2</sub>O) are all crystallising in beautiful octahedral crystals of pentagono-icositetrahedral symmetry. In the case of the chlorides and nitrates also hydrates with 3H<sub>2</sub>O were met with, which are formed in the solution at somewhat higher temperatures, and which possess a hexagonal-bipyramidal symmetry. Some Roentgen-spectroscopical investigations of the structure of these crystals will later-on be published elsewhere, which confirm these conclusions.

§ 4. The phenomena occurring in the preparation of the corresponding cobaltic salts, appear to be of an analogous nature. As the oxydation of a solution of CoCl<sub>2</sub> + Diaminocyclopentane with air gave no good

results, the formation of the cobaltic salt was performed by means of a 10% solution of *hydrogenperoxyde*: 7.5 Gr.  $\text{CoCl}_2$  in 25  $\text{cm}^3$  water were, in a wide flask, mixed with 5 Gr. *Diaminocyclopentane* and then 60  $\text{cm}^3$  of a 10% solution of  $\text{H}_2\text{O}_2$  were slowly added to it. When the reaction is finished, 100  $\text{cm}^3$  strong  $\text{HCl}$  are added to the solution and the liquid is then heated on the water-bath; soon the *praseo*-salt formed begins to precipitate.

The precipitate is then filtered off from the mother-liquid and so long washed with alcohol and ether, till complete neutral reaction is attained. The yield is about 6.2 Gr. of the *praseo*-salt; in the mother liquid there is still present some salt of the composition:  $\{\text{Co}(\text{Cptdine})_3\}\text{Cl}_3$ , which can be obtained by precipitation as *iodide*. In subsequently using in this reaction the racemic base or one of its optical-active antipodes, it seems to be possible to obtain the following salts:



These salts give, on heating them with a third molecule of one of the three bases mentioned, the desired *Tri-diaminocyclopentane-Cobaltic-salts*.

Both the last mentioned, bluish-green *praseo*-salts are optically-active; their molecules are different from their own mirror-images, but of course, they can individually not be separated. In Fig. 3 are reproduced the curves of rotatory dispersion of these *praseo*-salts, and that of the

Specific rotation  
in degrees :

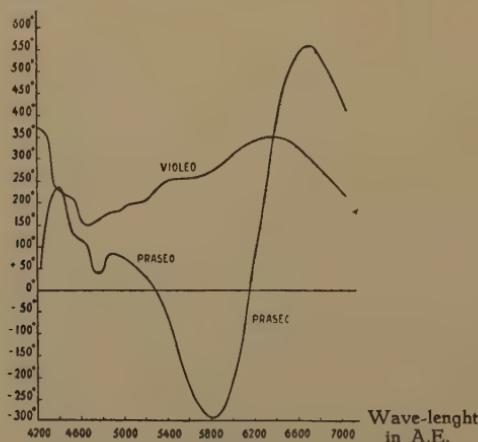


Fig. 3.

Rotatory dispersion of the *Praseo*- and *Violeo*- $\left\{ \begin{array}{l} \text{(l-Cptdine)}_2 \\ \text{Cl}_2 \end{array} \right\} \text{Cl}$ .

corresponding *violeto*-salts obtained by soft heating of the solutions of the first. The *praseo*-salt first mentioned above cannot be split into its antipodes, because its cation having a plane of symmetry perpendicular to the plane in which the two molecules of the base are situated. This salt, which is optically-inactive, must differ, therefore, from the true racemic compound obtained by mixing aequimolecular quantities of the second and third and by letting them crystallise from their mixed solutions. However, experience taught us, that this racemic compound could by no means be distinguished from the salt prepared from the racemic base; evidently the true racemic salt is likewise generated in this last reaction, so that the first of the three formulae mentioned above may be excluded beforehand.

Specific rotation  
in degrees

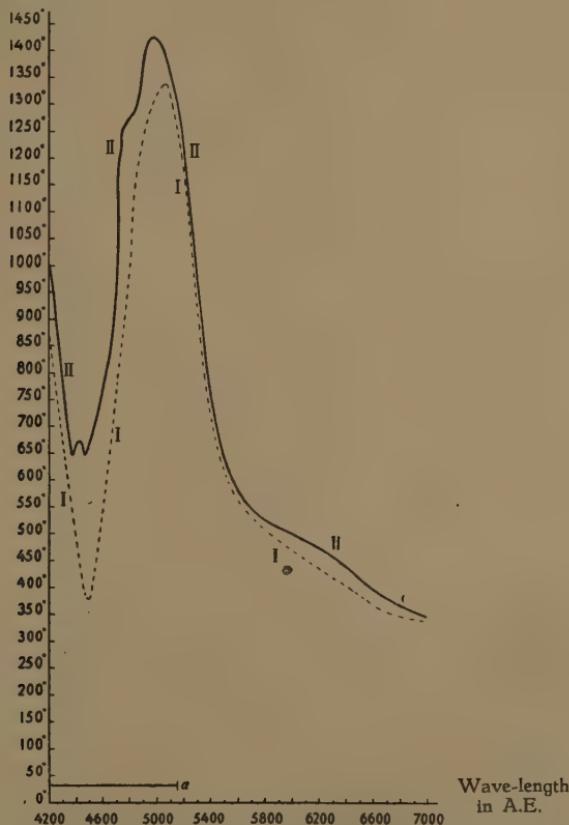


Fig. 4.

Rotatory dispersion of *d*-{Co(*l*-Cptdine)<sub>3</sub>}Cl<sub>3</sub> and its *Chloro-l-tartrate*.

By heating the *praseo*-salts with an excess of the base, we were able to obtain the corresponding *tri-diaminocyclopentane-cobaltic*-salts, as well in their racemic, as in their optically-active forms. In this case also, the racemic salt could be separated into its antipodes by means of the *chloro-d*-, and *chloro-l-tartrates*. The less soluble *chloro-d-tartrate* gave also in this case, — exactly as with the *Rho*-salts, — a *chloride*, which in solution was strongly *levorotatory*. It could be proved, by comparing its rotatory dispersion with that of the salts prepared from the active bases, that the constitution of its cation was:  $[ddd]_L$ . From this it becomes evident, that the behaviour of the *Rho*- and *Co*-salts is completely analogous; simultaneously the direct proof is given here of the view since long held by one of us<sup>1)</sup>, that the *Co*- and *Rho*-salts of the same spatial configuration show the *same* algebraic sign of their rotations, and *not*, as WERNER<sup>2)</sup> pointed out, opposite rotations. In the case of these *cobaltic* salts there also appeared to be formed only *two* kinds of complex cations with two active bases:  $[lll]_D$  and  $[ddd]_L$ , all other combinations being rigorously excluded by steric causes. Therefore, there is also only *one* racemic compound.

The curve of rotatory dispersion of these salts is reproduced in Fig. 4.

If in the *praseo*-salts of the types:  $\left[Co\frac{(d)_2}{Cl_2}\right]X_3$ , respectively  $\left[Co\frac{(l)_2}{Cl_2}\right]X_3$ , one tries to introduce a molecule of the *l*-, respectively of the *d*-base, in the hope of obtaining the salts of the type:  $\left[Co\frac{(d)_2}{(l)}\right]X_3$  and  $\left[Co\frac{(l)_2}{(d)}\right]X_3$  respectively, — it appears, that this does not happen. The expected combinations seem to be unstable from steric causes, and only a mixture of 1 molecule of an active salt with three identical molecules of the base in its cation, besides 1 molecule of the racemic compound is obtained in each case, according to the reaction-scheme:



In this case also the formation of the cations with the *highest* possible symmetry is evidently favoured at the cost of the complexes of a lower degree of symmetry.

§ 5. In general the crystalforms of the racemic and optically-active *Co*- and *Rho*-salts appear to be quite analogous. The optically-active components here also crystallize in octahedral crystals of pentagon-icositetrahedral symmetry. In the case of the *chloride* and of the *nitrate*, there exist, here also, besides the cubic forms, hexagonal-bipyramidal hydrates, crystallizing with  $3 H_2O$ .

Moreover, the corresponding *perchlorate* shows also in this case the

<sup>1)</sup> F. M. JAEGER, Proceed. Royal Acad. Amsterdam, **20**, (1917) 264, 265. Chem. Weekbl. **14**, (1917), 728; Recueil d. trav. chim. Pays-Bas, **38**, (1919), 170; *Lectures on the Principle of Symmetry*, 2nd Edit. (1920), p. 251.

<sup>2)</sup> A. WERNER, Ber. d. d. chem. Ges., **45**, (1912), 1229.

phenomenon of a spontaneous fission into its antipodes by crystallisation at room-temperature. This is most remarkable, because circumstances are here just the same as those in the case of the *Rho*-salts: the transformation-temperature, — if really existing, — must be here also a *minimum* temperature in the neighbourhood of  $-1^{\circ}5$  C., as is clearly demonstrated by the results of the determinations of the solubilities (see Fig. 5). A direct measurement of this transition-temperature is impossible, because at those low temperatures the solubilities have become already so very small, that the actual depression of the freezing-point is too insignificant, to retain under these conditions the appearance of a liquid solution besides the solid phase; all is here already converted into a solid mixture of ice and salt.

But in every case, we must conclude that at all temperatures above

Solubility in grammes  
anhydrous salt in  $100\text{ cm}^3$ . sol.

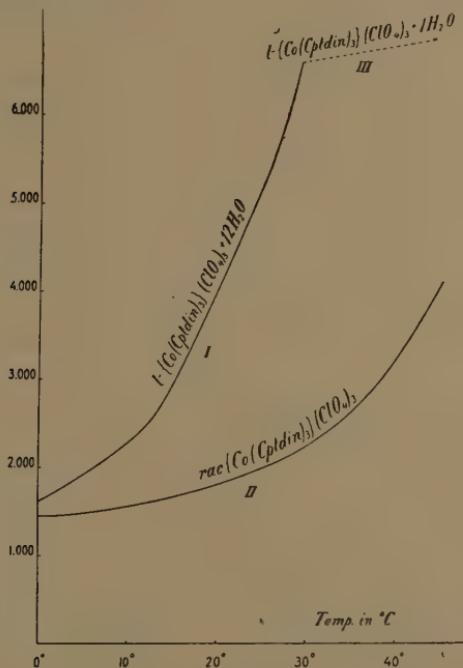


Fig. 5.

Solubility of rac. and levo-rot.  $\{\text{Co}(\text{Cptdine})_3\}(\text{ClO}_4)_3$  at different temperatures.

$0^{\circ}$  C., the optically-active, octahedral crystals must be *metastable* with respect to the racemic compound. Therefore, it is very remarkable that, notwithstanding this, these octahedral crystals are yet deposited in the inactive solution at room-temperature. That the components are really

metastable with respect to the racemic compound under these conditions, can be also seen from the fact, that if the saturated solutions be inoculated with a germ of the crystallized racemic salt, the octahedral crystals will gradually disappear in favour of those of the racemic compound.

Moreover, the colour of these *Co*-salts is highly remarkable: they being clear *pink*, as true *cobalto*-salts in solution, instead of having the deep orange or bloodred colour of the *triethylenediamine-cobaltic*-salts. The *iodide*, however, is an exception to this, showing the true colour of the *luteo*-series.

The specific and molecular rotations of these *Co*- and *Rho*-salts are, moreover, for the same wave-lengths many times greater than those of the corresponding *triethylenediamine*-salts: the absolute values of the specific rotations are, in the case of the *cobaltic*-salts, about 3 to 10 times, in the case of the *Rho*-salts about 7 to 8 times greater than they are in the corresponding salts of the *triethylenediamine*-series. The rotatory dispersion of the *Rho*-salts, on the other hand, does not differ in both cases to the same degree as the rotations themselves.

§ 6. Much more complicated, however, are the phenomena observed, if besides *Diaminocyclopentane*, one tries to get complex salts of *cobaltum*, whose cations do also contain 1 or 2 molecules of *ethylenediamine*.

For the preparation of salts of the type:  $\left\{ \text{Co} \begin{smallmatrix} (\text{Eine})_2 \\ (\text{Cptdine}) \end{smallmatrix} \right\} \text{Cl}_3$ , we can start with the green *praseo*-, or the pink *chloro-aquo-cobaltic*-salts containing 2 molecules of *ethylenediamine* in their complex ions.

If e.g. salts like:  $\left\{ \text{Co} \begin{smallmatrix} (\text{Eine})_2 \\ \text{Cl}_2 \end{smallmatrix} \right\} \text{Br}$  or  $\left\{ \text{Co} \begin{smallmatrix} (\text{Eine})_2 \\ \text{Cl} \\ \text{H}_2\text{O} \end{smallmatrix} \right\} \text{Br}_2$  are treated with racemic or *d*-, and *l*-*Diaminocyclopentane*, there are really formed salts of the type desired. The simplest case is that, where 1 molecule of the *racemic* base is used. An optically-inactive salt, having the colour of the ordinary *triethylenediamine-cobaltic*-salts, is produced in this case, which appears to be a true *racemic* compound, as is proved by its possible fission into two optically-active antipodes by means of the *bromo-d*-, respectively the *bromo-l-tartrates*. In fig. 6 is reproduced the dispersion-curve (II) of the *bromide*, prepared from the less soluble *bromo-d-tartrate*; this *bromide* was proved to have the constitution:  $\left\{ \text{Co} \begin{smallmatrix} (\text{Eine})_2 \\ (\text{l-Cptdine}) \end{smallmatrix} \right\} \text{Br}_3 + 2 \text{H}_2\text{O}$ . The other component gave the same rotations, but with opposite signs.

The difference in the character of this dispersion-curve with that of the salt of the ion:  $\left\{ \text{Co}(\text{l-Cptdine})_3 \right\}_D$  is very clear, as may be seen by comparing Fig. 4 and 6 with each other. The curves have in common only the high peak near their absorption-band, which in the case of the last mentioned salts is situated at about 4950 A.U., in the case of the

newly prepared  $\left\{ \text{Co} \left( \text{Eiine} \right)_2 \right\}_{\text{D}} \left( \text{l-Cptdine} \right) \left\{ \text{Br}_3 \right\}_{\text{D}}$ , at about 4800 A.U. But the most

Specific rotation  
in degrees

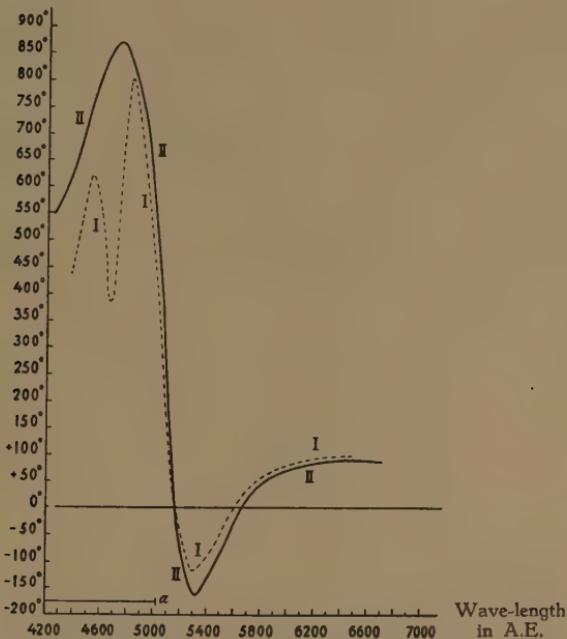


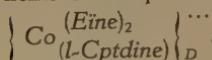
Fig. 6.

Rotatory dispersion (II) of  $d\text{-} \left\{ \text{Co} \left( \text{Eiine} \right)_2 \right\}_{\text{D}} \left( \text{d-Cptdine} \right) \left\{ \text{Br}_3 \right\}_{\text{D}}$  and its Bromo-d-tartrate.

typical difference between the two curves, is the twice passing of the axis of zero-rotation in the last case, at about 5200 and 5700 A.U.

§ 7. If now, however, instead of the *racemic* base, the *levorotatory* component, for instance, be used in the reaction indicated above, then there results an optically-active product, which shows *quite another rotatory dispersion* than that reproduced in Fig. 6. In the first place its specific rotation appears to be much less: the curve in Fig. 7 indicated with *I*, gives the observed values in graphical form. (See Fig. 7). On closer examination it was found, that here there is formed a kind of optically-active *partially racemic* compound, which consists of two optically-active components, which are, however, *no longer mirror-images* of each other. We succeeded in separating from this product, by the aid of *d-tartaric acid*, a *bromo-d-tartrate*, whose rotatory dispersion appeared to be the

same as that of the *bromo-d-tartrate* prepared from the *bromide*, represented in Fig. 6. From this it becomes certain, that one of the components of the partially racemic compound is a salt derived from the ion:



Although it appeared impossible, because of the very unfavourable solubility-relations of both components, — even if other substances, as,

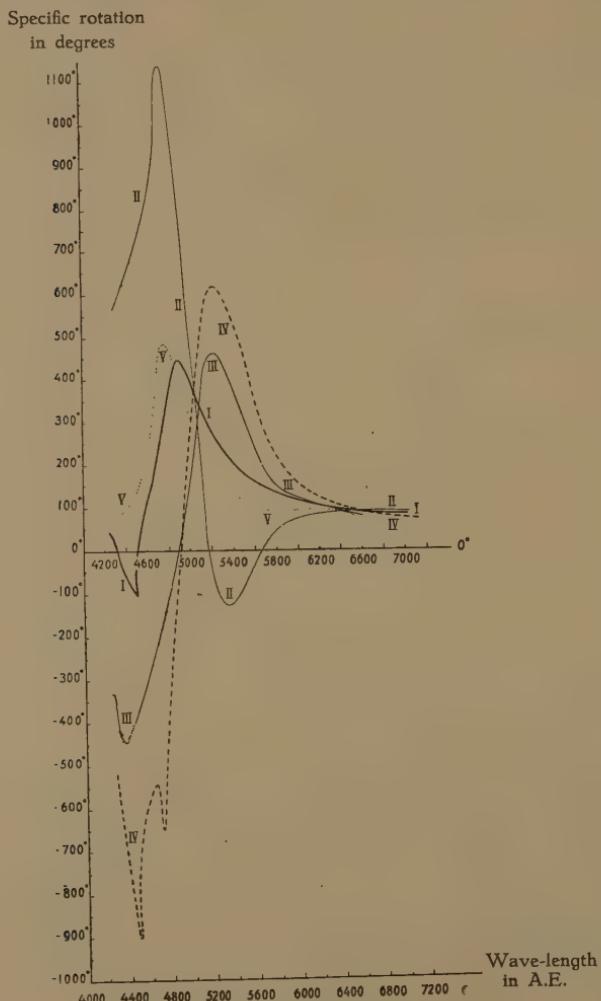


Fig. 7.  
Rotatory dispersion of the part. racemic  $\left\{ \text{Co} \begin{smallmatrix} (\text{Eine})_2 \\ (\text{l-Cptdine}) \end{smallmatrix} \right\} \text{Br}_3$  and its components.

for instance, *mandelic acid*, *malic acid*, *bromocamphersulfonic acid*, *α-nitrocamphor*, etc., were used in these experiments, — to obtain the second component in a perfectly pure state, there can be no doubt whatsoever about the fact, that the other component is a derivative of the ion:  $\left\{ \text{Co}^{(\text{Ei}\text{ne})_2}_{(\text{L-}\text{Cptdine})} \right\}_L^{\cdots\cdots}$ , which, indeed, is *not* the mirror-image of the first.

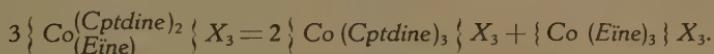
In Fig. 7 we have reproduced sub III the observed curve of rotatory dispersion of this yet impure, second component, while curve IV should represent the theoretically calculated dispersion-curve of this same component in the pure state. This was determined by combination of the curves I and II, of which the last is the true dispersion-curve of the pure first component. Finally the curve V of Fig. 7 represents the curve of rotatory dispersion of the partially racemic compound, as it follows from the combination of the curves II and III: the analogy with curve I is evident, although the inaccuracies in the curve III cause indubitable deviations from the true form of I.

In the same way the reaction between *d-Diaminocyclopentane* and the original *praseo-* (*violet*-)-salt, gives a partially-racemic compound, consisting of salts derived from the cations:



If it would have been possible for us to obtain each of these four components in a perfectly pure condition, we would have been able to obtain, besides four partially racemic compounds also the two optically-*inactive*, true racemic compounds, by combining the antipodes in pairs in *aequimolecular* quantities, and a racemoid, containing all four components.

§ 8. To prepare the salts of the type:  $\left\{ \text{Co}^{(\text{Ei}\text{ne})}_{(\text{Cptdine})_2} \right\} \text{Cl}_3$  we started from the *praseo-* (*violet*-)-salts of the formula:  $\left\{ \text{Co}^{(\text{Cptdine})_2}_{\text{Cl}_2} \right\} \text{Cl}$ , described in the above. With these salts 1 molecule of *ethylenediamine* was made to react under the most varied circumstances of temperature, concentration and in different solvents; but it became soon clear that the salts of the type desired were *not* sufficiently stable to obtain them from solutions, a reaction of the form:



occurring in each case. In some experiments, — e.g. during our attempts to split the reaction-product into two optically-active components, — we got some indications of the fact, that the salts of the type desired were actually formed as intermediary products; but on recrystallizing the impure mass from solutions, we never obtained other substances than a

mixture of the last mentioned kinds of complex salts. The components were identified in this case by their precipitation as *iodides*, and by accurately studying their crystal-forms.

The manner of transformation in this case, is, once more, perfectly analogous to that formerly observed in the transformation of the cobalt-salts of the type cation:  $\{\text{Co}(\text{ddl})\}^{\cdots}$ . In both cases it is always the type of ion with a lower degree of symmetry (with one single, polar binary axis), which is replaced by a mixture of two kinds of cations possessing both a much higher symmetry ( $D_3$ ). One of us<sup>1)</sup> has already drawn attention on a former occasion to the special tendency towards the formation of such complexes of a higher degree of symmetry. The facts stated here give a convincing confirmation of the truth of the views then stated with respect to this matter. It is the significance of the maximum symmetry in questions of the relative stability of such spatial arrangements, that in this case also reduces the number of the theoretically possible isomerides of such complexes to such an extent as was observed here.

§ 9. The experiments with salts of the type:  $\left\{ \text{Co} \begin{smallmatrix} (\text{Eine})_2 \\ \text{Cl}_2 \end{smallmatrix} \right\} \text{Cl}$ , described

in this paper, prove, however, clearly that the introduction of a molecule of an optically-active base, e.g. of the *levorotatory Diaminocyclopentane*, can occur as well in a *dextro*-, as in a *levorotatory* configuration of a cation, in which 2 molecules of *ethylenediamine* are already present. This introduction is, however, impossible, if instead of 2 molecules of *ethylenediamine*, there are originally present, 2 molecules of the optically-active antipode of the substituent. From this it appears once more, that the contrast between a *d*- and *l*-isomeride of the same compound appears to be much more pronounced than that between an optically-active molecule and totally strange substances in those cases, where attempts are made to introduce them all together into the same dissymmetrical spatial configuration. The preliminary presence of such strange molecules, appears to be a much less severe impediment to the entrance of an optically-active substituent, than is produced by the presence of molecules of the same kind, but with enantiomorphic arrangements; — a fact which reminds us altogether of the well-known highly „specific” action of optically-active ferments and enzymes on attackable substrates of enantiomorphous configurations.

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<sup>1)</sup> F. M. JAEGER, Proceed. Kon. Akad. v. Wet. Amsterdam, 29, (1926), p. 575—579.

**Chemistry. — On the Symmetry and Structure of the cubic Nitrates of Calcium, Strontium, Barium and Lead.** By Prof. Dr. F. M. JAEGER and F. A. VAN MELLE.

(Communicated at the meeting of June 30, 1928).

§ 1. Between the results of the crystallographical and the ROENTGEN-spectrographical investigations on the true symmetry of the isomorphous series of cubic *nitrates* of the metals *calcium*, *strontium*, *barium* and *lead*, there are still a number of discrepancies, which cause the question to be quite unsettled.

As far as concerns the geometrical form of these crystals, all data obtained by the different crystallographers seem to indicate, that the symmetry is truly the *tetrahedral-pentagonedodecahedral* one. The *Ca*-salt was obtained by RETGERS<sup>1)</sup> in apparent octahedra from the monoclinic *tetrahydrate* on heating it at 150° C.; it is isomorphous with the *Sr*-salt, which also forms a monoclinic *tetrahydrate*, but which may also be obtained directly in an anhydrous cubic form, if crystallized from warm solutions. WULFF<sup>2)</sup> showed, that in most cases it crystallizes in apparent *cubo-octahedra*, but that its twins often manifest also a *dodecahedron* {210}. SCACCHI<sup>3)</sup> stated, that a slight addition of  $Fe_2Cl_6$  to its solutions, favours the development of this last form, and WULFF found the same fact, if  $CaCl_2$  be added to the solution.

According to this author, the *Ba*-salt shows, besides both *tetrahedra* {111} and {111} and the *cube* {100}, moreover a *pentagonal dodecahedron* {201} and a left-handed, positive *tetrahedral pentagonedodecahedron*, also in the case that no strange salts are added to the solutions. The development of this last form is favoured by an addition of  $NaNO_3$  to its solution; while the presence of  $KNO_3$ ,  $NaNO_3$  and *saccharose* causes the appearance of the forms {211} and {221}, and that of *ferric nitrate* leads to the manifestation of {883} and {511}. LEWIS<sup>4)</sup> observed the *tetrahedral pentagonedodecahedra*: {351}, {214} and {214}, in addition to the usual limiting forms; and HENRIQUES<sup>5)</sup> showed, that on addition of some *nitrophenol* to the solution, the *pentagonal dodecahedron* {201} becomes the predominant form. Although the study of the etching-figures has also led to results<sup>6)</sup>, which agree with the symmetry mentioned above, they must now be considered as having only a feeble

<sup>1)</sup> J. W. RETGERS, Zeits. f. Kryst., **21**, (1893), 257.

<sup>2)</sup> L. WULFF, ibid., **4**, (1880), 139.

<sup>3)</sup> A. SCACCHI, Pogg. Ann. der Physik, **109**, (1860), 366.

<sup>4)</sup> W. J. LEWIS, Zeits. f. Kryst., **2**, (1878), 64.

<sup>5)</sup> R. HENRIQUES, Ref. ibid., **5**, (1881), 365.

<sup>6)</sup> H. VON BAUMHAUER, Z. f. Kryst., **1**, (1877), 52. On {100} etching-figures were observed which correspond with a left-handed negative *tetrahedral pentagonedodecahedron*; also WULFF (*loc. cit.*) mentions the same fact.

conclusive force, because of the fact that in recent times it could be proved, that the presence of slight traces of impurities can have a strong influence upon this phenomenon in the case of  $KCl$ ; in the older investigations this fact could, in general, not yet be sufficiently taken into account.

In the case of  $Pb(NO_3)_2$  WULFF<sup>1)</sup> observed, on rapid evaporation of the solution, the tetrahedra  $\{111\}$  and  $\{111\}$  in unequal development, besides  $\{883\}$  and a tetrahedral pentagonedodecahedron, which approximately seemed to correspond with the symbol:  $\{10.5.6\}$ . According to this author, the appearance of this last form seems to be favoured by the presence of some  $Zn(NO_3)_2$ . An analogous influence was observed by SCACCHI with respect to the appearance of  $\{201\}$ , if traces of  $NaNO_3$  were added to the solution. It may be remarked here, that we have observed this pentagonal dodecahedron also, in a well-developed form, if no other salts were added to the solution. According to MOREL<sup>2)</sup>, the form  $\{201\}$  would be favoured also by the presence of  $HNO_3$ .

As is well known, MARBACH as the first observed, that all these *nitrates* show an abnormal birefringency, which in the case of the *lead-salt*, for instance, is particularly strong, and which is studied several times in detail. It manifests itself by a division in optically-abnormal sections of the crystals and is often explained by the presence of internal strains in the crystalline individuals.

§ 2. On the contrary, there are also facts, which seem to prove against the tetrahedral-pentagonedodecahedral symmetry of these crystals. There is, for instance the peculiar circumstance, that never even a trace of circular polarisation could be detected with them, — not even in LONGCHAMBON's more recent and accurate investigations. Another difficulty is presented by the investigations of their pyro-electrical behaviour by HANKEL, who says that the forms  $\{111\}$  and  $\{111\}$  may be discerned by their opposed pyro-electric polarity, — a fact which in the cubic system is, however, impossible<sup>3)</sup>. On the other hand, no piezo-electrical phenomena are, up till now, observed with these crystals<sup>4)</sup>.

But also from the results of the ROENTGEN-spectrographical study of these *nitrates* the conclusion seems to become evident, that they have another symmetry, the *dyacisdodecahedral* one. Besides some ROENTGEN-spectrographical observations made by NISHIKAWA and HUUNIKI<sup>5)</sup> and some not published results of WYCKOFF<sup>6)</sup> obtained by means of LAUE-spectrograms, there are also calculations of the intensities made by

<sup>1)</sup> L. WULFF, *loc. cit.*

<sup>2)</sup> J. MOREL, *Bull. de la Soc. minér.* **9**, (1886), 294.

<sup>3)</sup> E. RIECKE, *vid. L. GRAETZ: Handbuch der Elektrizität und des Magnetismus*, Bd. **1**, (1918), 407.

<sup>4)</sup> Conf. e.g. P. TERPSTRA, *Natuurw. Tijdschrift*, **10**, (1928), 92; *Congres Natuur- en Geneeskunde*, Leuven, (1928).

<sup>5)</sup> S. NISHIKAWA and K. HUUNIKI, *Proceed. Math. Phys. Soc. Tokyo*, **9**, (1917), 197.

<sup>6)</sup> R. W. G. WYCKOFF, *The Structure of Crystals*, (1924), 353.

VEGARD, Zeits. für Physik **9**, (1922), 395, from powder-spectrograms after HULL-DEBIJE. They all seem to lead to the conclusion, that the symmetry is *dyacisdodecahedral* and that the space-group is  $T_h^6$ . In the presence of such strongly diffracting atoms like those of *Ba* and *Pb*, such a calculation of the diffraction-intensities from powder-spectrograms has no significance whatsoever with respect to the placing of the  $(NO_3)$ -groups within the primary cell. To the space-group  $T_h^6$ , it is true, corresponds the simple cubic lattice; but no diffraction-images were found, which would be incompatible with those of a substance having a face-centred cubic lattice, because the *Ba*- and *Pb*-atoms which in the last case would have the positions: [000],  $[0 \frac{1}{2} \frac{1}{2}]$ ,  $[\frac{1}{2} 0 \frac{1}{2}]$  and  $[\frac{1}{2} \frac{1}{2} 0]$ , are so predominant in their diffracting power, that no difference in both case is detectable. However, in the case of powder-spectrograms the choice between the groups  $T_4$  and  $T_h^6$  must be made by looking for the absence of reflections of the first order of  $\{hko\}$ , if  $h$ , as well as  $k$  are both *odd*. Now the predominant action of the *Ba*- and *Pb*-atoms in the corresponding *nitrates* already mentioned as well as the fact, that with *Ca*- and *Sr-nitrates* only a single diffraction-line of  $\{hko\}$ , namely only (210), was observed, — makes it quite impossible to fix a choice between  $T_4$  and  $T_h^6$  by means of powder-spectrograms. It was, therefore, necessary to repeat these investigations, making use this time of rotation-spectrograms.

§ 3. Rotation-spectrograms were made by means of an apparent cubo-octahedron of  $Sr(NO_3)_2$ , the directions [110] and [100] being used as axes of rotation. Figure 1 shows the spectrogram obtained with [110]



as axis of rotation, the mean position being such, that it made  $15^\circ$  with (001) and the crystal being rotated through  $10^\circ$  to both sides of it. From the reciprocal lattice it can be deduced, that (120), (201), (210), (021), (110), (320), (330), and (100) will be in such positions, as to give the corresponding reflections. It appeared, that (210) and (021) were truly *present*, but that (120), (201), (110), (320) and (330) were *lacking*. In the case of the rotation-spectrogram round [100], the exposition was 3 hours, while the rotation occurred in such a way, that eventually (003) and (005), respectively (005) and (007) would possibly appear. Although the cube-face used was beautifully developed and very smooth, the said reflections of odd orders appeared, however, to be wholly *absent* in each case.

From these observations the conclusion must be drawn, that the fundamental lattice is the *simple cubic* one, with:  $a_0 = 7.80$  A.U. and with 4 molecules  $Sr(NO_3)_2$  in the elementary cell. Because, moreover, the first order of  $\{hko\}$ ,  $h$  being *odd*, is *not* present here, this fact points to the space-group  $T_h^6$ .

§ 4. LAUE-spectrograms were now made in the case of *Sr*-, *Ba*-, and *Pb-nitrate*. (Tungsten-anticathode; 50 K.V. tension). It appeared that the spectrograms thus obtained by radiation  $\perp$  to  $\{111\}$ , did *not* possess symmetry-planes passing through the trigonal axis; — a result, evidently quite in accordance with the unpublished results of WYCKOFF (conf. loco cit., p. 353). The LAUE-spectrograms of the three *nitrates* mentioned, showed the greatest possible analogy; it is most remarkable, that they do almost not deviate from those of a true holo-cubic crystal (three spots are the only ones, which are *unsymmetrically* situated) with three symmetry-planes  $\perp$  (111).

A rotation-spectrogram of  $Ba(NO_3)_2$  with [100] as axis, proved again, that the reflections (001) and (003) were both *absent*; but there appeared here also a reflection of (210) in the 1<sup>st</sup> order, while (120) was *absent*. Therefore, it must be concluded from this, that the space-group of  $Ba(NO_3)_2$  is also  $T_h^6$ , with:  $a_0 = 8.10$  A.U. and with 4 molecules of the salt in the elementary cell.

§ 5. Now a rotation-spectrogram of  $Pb(NO_3)_2$  was made with [100] as axis of rotation, the primary position of the crystal being such, that it was at  $36^\circ$  with (010), while the rotation occurred through an angle of  $10^\circ$  to both sides, i.e. from  $26^\circ$  to  $46^\circ$  with respect to (010). In this case (003), (005) and (007) had an opportunity to reflect. But, although the crystal used had a beautiful, smooth face of several  $mm^2$  surface, *no other* reflections than that of (004) and (006) could be obtained, not even after prolonged exposition: also in this case all reflections of *odd* order appeared to be *absent*.

BRAGG-spectrograms, obtained by exposition during several hours and by rotating through a small angle in such a way, that the images of (001), (002) and (003) could be expected, gave *no* images of any appreciable intensity, save (002), the others having intensities certainly less than

$\frac{1}{10}$ th part of that of (002). Also from these data it becomes almost certain, that the space-group is  $T_h^6$ , having thus *dyacisdodecahedral* symmetry.

On (210) a BRAGG-spectrogram gave the 1<sup>st</sup> order image of (210); from our spectrograms, — calcite being used for the purpose of comparison, — the value of  $a_0$  was deduced to be: 7.80 till 7.81 A. U.  $\pm 0.01$ , in stead of VEGARD's value: 7.84 A. U.

§ 6. The attempts to fix the positions of the N- and O-atoms in this case by a calculation of the intensities, can only have any significance in the case of the Ca-salt. In that of the Ba- and Pb-salts it would be only possible perhaps to obtain better information about the true positions of these atoms by means of accurate measurements with an ionisation-spectrometer; but it is excluded in these cases to obtain any reliable results from powder-spectrograms.

Better than powder-spectrograms, these LAUE-photo's indicate the severe difficulties present here for the calculation of the four parameters of the structure. For, as the great majority of the diffraction-spots is compatible with the holo-cubic symmetry, — even in the case of  $\text{Sr}(\text{NO}_3)_2$ , where the number of electrons of the metal is about the same as that of the  $(\text{NO}_3)$ -groups, — it is highly probable, that the oxygen-atoms *approximatively*, and the metal-atoms *rigorously*, have positions in the structure of these crystals, which are compatible with those present in one of the classes of holo-cubic symmetry (for instance, position 24<sup>a</sup> of WYCKOFF's list). The possibility seems, therefore, excluded beforehand, of fixing the true positions of the N-atoms with their relatively small number of electrons, in the midst of the seven times greater number of the electrons of the other constituents of these salts. Exactly because the method based upon the diffraction of ROENTGEN-rays is still a method of small sensitivity, as regards the differences between heavier and lighter-atoms, diffère nécessairement de l'ensemble de tous les points de  $R$ , de sorte que l'ensemble fermé  $F_\alpha = R - \sum_{\beta=1}^{\alpha} G_\beta$  est non vide, quel que soit  $\alpha > \Omega_r$ . — the possibility, that yet a tetrahedral-pentagonedodecahedral arrangement with the symmetry  $T_4$ , — but *very nearly approaching* to an arrangement with the symmetry  $T_h^6$ , — may be present here, needs still to be considered in future: the facts revealed by all crystallographers seem, as was pointed out before, to indicate the symmetry of the group  $T$ .

Certainly it is necessary to investigate the complete structure of these nitrates in detail. For, notwithstanding the results obtained here with respect to the symmetry and space-group, a number of physical discrepancies remain.

In every case VEGARD's conclusion about a *dyacisdodecahedral* symmetry can now be considered as being right, notwithstanding the fact that *no proof* was given by him. The further investigation of the structure of these nitrates can be continued now, starting from the data here obtained.

Groningen, Laboratory for Inorganic and  
Physical Chemistry of the University.

**Medicine. — On the Nature of Cardiac affection in Beriberi Patients.**  
By Prof. K. F. WENCKEBACH and W. C. AALSMEER.

(Communicated at the meeting of February 25, 1928).

Accurate and exhaustive clinical investigations by DR. AALSMEER, Lecturer on Internal medicine at the Surabaja Medical School for Indian Physicians (N. I. A. S.) have enabled us to find an explanation for the puzzling, most uncommon course of the perilous heart-trouble in cases of Beriberi. The body of facts now on hand proves that an increasing weakness of the cardiac contraction involving the whole heart, is the cause of a considerable disturbance of circulation. The latter will be discussed at large in the Ned. Tijdschr. v. Geneeskunde. In the present paper we will only emphasize some points that will help us in identifying the nature and the course of the disease itself.

These points are:

1. An increased thickness of the cardiac wall and an increase of the weight of the entire organ, considered together as a "hypertrophy". These changes appear already at the beginning of the disease on the left, as well as on the right half of the heart and are independent from the nervous degeneration occurring in Beriberi-patients.

2. The electro-cardiographic investigation shows that, in spite of the largely decreased strength of the heart, the automatic processes of the formation of stimuli, excitability, and stimulus-conduction, are completely normal.

3. The utter failure of all heart-tonics in use (digitalis, strophanthus, coffein and theobromin preparations), contrasted with the complete disappearance of all cardiac phenomena after the administration of vitamin B (Katjang Idjoe etc.).

I. It is difficult to assume in one and the same heart-muscle the existence of considerable, nay, fatal weakness, and at the same time a true hypertrophy of the muscle, which is indicative of stronger activity. A hypertrophic muscle may, indeed, fail, but it cannot be expected that a muscular activity that has been enfeebled from the onset, should induce hypertrophy.

A second and in fact insuperable objection to the assumption of a hypertrophy, is the long known fact, in every respect confirmed by AALSMEER, that the administration of a vitamin-containing diet abolishes the considerable enlargement of the heart within a short time in very severe cases. It is not probable, however, that an existing hypertrophy would not leave

traces after treatment, so that the only alternative is to find other causes for the enlargement of the heart and the increase of its weight than those prominently brought forward by all earlier authors.

II. In our search for these causes this point has perhaps led us on the right track. Many years ago ENGELMANN reported, at a meeting of this Academy that the heart-muscle, just as the skeletal muscle (BIEDERMANN), swells in water and loses its contractility, whereas excitability and conductivity persist all the same. Many years later DE BOER pointed out that the swelling itself hampers the contraction of the muscle fibre mechanically, but that in this process the E. C. G. remains perfectly normal. That the same phenomenon, a weakening of the force of contraction and normal E. C. G., was found in Beriberi-patients, suggested to us the idea that here also disturbance might be caused by absorption or retention of water ("Quellung").

Potent arguments lend support to this assertion. According to TIEMANN's<sup>1)</sup> recent inquiries all striated muscle, contrary to smooth muscle, swells in every hypotonic fluid examined by him, the cardiac muscle less than the skeletal muscle. If a similar process exists in Beriberi, such swellings may also be expected in skeletal muscle. In fact they belong to the typical aspect of Beriberi, already strongly pronounced at the calf muscle at the beginning of the disease, even where edema of the feet and symptoms of degeneration of the nerves cannot yet be demonstrated. This swelling may be identified by the pain caused by pinching the calf, which is so hard and enlarged, that in MENSE's Lehrbuch der Tropenkrankheiten NOCHT compares it with the well-known "pseudohypertrophy" of the progressive muscular atrophy in children (fig. p. 461). PEKELHARING and WINKLER already speak of a *seeming* muscular hypertrophy.

III. The next point also furnishes signal evidence for the explanation given here. It is easily understood that only such a treatment can exert a favourable influence upon the Beriberi-heart which is able to draw the absorbed water from the cardiac muscle. That digitalis cannot do this could be expected. However, it is very remarkable that no influence whatever is exerted on the heart by diuretics, neither by thyroidpreparations nor by a treatment expelling water from the tissues so potently as the mercury-preparations novasurol and salyrgan. (AALSMEER, SCHIMAZONO).

In contrast with this, vitamin B (whose absence evokes all the symptoms of Beriberi) brings on an almost marvellous improvement shortly after it is given to the patient. The literature shows conclusively that not only does the cardiac muscle resume its contractility together with, in our opinion

<sup>1)</sup> F. TIEMANN Ueber die Quellungsbedingungen des Herzmuskels unter verschiedenen Bedingungen. Ztschr. f. d. ges. exper. Medizin. 58, 821, 1928.

also by virtue of its smaller size and weight, but that also the skeletal muscles are benefited by this influence. The firm, thickened and painful muscles are immediately reduced to their normal size and function. In his monograph on Beriberi, published last year, DE LANGEN, when alluding to the disappearance of the edema, speaks of "the still more remarkable quick recovery of the pareses, and sometimes already paralyses, which within 24 hours were things of the past after the diet had been duly modified." According to SCHEUBE<sup>1)</sup> as early as 808 B. C. Katjang Idju was known in Japan to be "ein harntreibendes Mittel". Also in later publications this property of vitamin-treatment is often recorded. That this substance acts so differently from the ordinary diuretic treatment, favours the conception that the swelling of the striated muscles is due to a specific cause. This fact may perhaps supply an argument in answering the question whether and how far this swelling is of an infiltrative, osmotic or colloidal nature.

It would seem then that our hypothesis of the absorption of water by the heart-muscle being the causative factor, affords an explanation of the whole set of symptoms of the syndrome of Beriberi, most unique in cardiac Pathology. It seems altogether probable that other manifestations of this disease, notably the degeneration in the central and peripheral nervous system, are also due to the same cause, viz. swelling by retention of water. A good many data taken from the experimental literature seem to lend support to this view. Further investigations will have to solve this and many other problems. Likewise it may be deemed probable that analogous, or even identical processes come into play in certain other forms of insufficient action of the heart. The often incomprehensible total absence of response of the diseased heart to all our heart-stimulants, which is generally indicative of a hopeless condition of the ventricular muscle, may in some cases be explained on the same line. In this connection one is reminded of myxoedema and its treatment by the administration of thyroid, which substance, however, does not seem to act in cases of Beriberi.

There are still other reasons for giving attention to the subject under discussion. It lays a bridge between scientific clinical examination and the experimental work on problems that have recently come to the front, viz. those of the so-called "heart-hormones" and "automatins". The fact that the organism itself produces substances that can revive a heart after it has been brought experimentally to a standstill, and seem to be able to restore the automatic function, is recommended rather inconsiderately for the treatment of heart-patients. This is certainly premature, but anyhow the Beriberi heart shows that the revival of a heart, given up for lost, is not out of the bounds of probabilities. The communication made by ZWAARDEMAKER at the previous meeting of this Academy, is also of vital importance. The radiated vitamin B is the same substance that can save a

<sup>1)</sup> SCHEUBE, *Die Beri-beri*. Fischer 1894.

Beriberi heart. It is said to be identical to histamin, which according to inquiries published in Vienna a few days ago, is contained in various substances regarded as heart-hormones.

Meanwhile the Beriberi heart affords an instructive example for thorough investigation ; it reminds us in good time of the fact, sometimes forgotten in the hormone literature, that not every arrested heart has altogether lost its regular automatic vital function. Automatism may continue undisturbed, but in consequence of the swelling of the muscle-fibers hindrance, or other may not be able to manifest itself in a visible form. The added active substance should, therefore, in such a case not be called automatin ; what takes place in the heart is not a resuscitation from the sleep of death, but the deliverance of a fettered prisoner.

*Vienna, 18 February 1928.*

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## Geology.—*Die Laven vom Etna.* By W. F. GISOLF.

(Communicated at the meeting of May 26, 1928).

Die von mir eingeführte Projektionsmethode<sup>1)</sup> des Niggliischen Tetraeders ist besonders geeignet zum Studium der Gesteine einer petrographischen Provinz und der Gesteine, welche im Laufe der Jahrhunderte von einem Vulkan oder einer Vulkangruppe zu Tage gefördert worden sind. Die genannte Methode hat den sehr wesentlichen Vorteil, dass man nicht nur im Stande ist, in übersichtlicher Weise der Zusammensetzung der wirklichen Gesteinsmineralien Rechnung tragen zu können, sondern auch, dass Beziehungen zu Tage treten, welche man sonst gar nicht oder nur auf umständliche Weise hätte finden können; vorausgesetzt, dass man zuvor nicht den Nutzen solcher endlosen Berechnungen angezweifelt hätte.

Als treffendes Beispiel habe ich für diesen Artikel die Gesteine gewählt, welche aus demjenigen Magma ans Tageslicht getreten sind, das den Etnavulkan auf Sizilien aufgebaut hat; Gesteine mit einer ziemlich gleichförmigen Zusammensetzung, deren Normberechnung nur einen geringen Einblick ergibt in dem Charakter des Magmas, dessen Kristallisation diese Gesteine ihr Dasein verdanken.

Die Laven von Sizilien fangen in der Vorkreide- oder in der Kreidezeit an auszufließen in der Nähe von Capo Passero, in der Südecke der Insel. Die Analysen dieser Gesteine (2 an Zahl) habe ich nicht zu Rate ziehen können; was vermutlich nicht schlimm ist, da sie um 1881 ausgeführt worden sind. Auf diese Eruptionen folgten im Miozän diejenigen des Val di Noto, etwa 60 km nordöstlich von Capo Passero. Von diesen Gesteinen sind einzelne gute Analysen bekannt, deren drei in Tabelle I aufgenommen sind (Nº. 6, 23, 24). Darauf sind in Gebiete, wo nachher der Vulkan sich erheben sollte, in quaternärer Zeit Gesteine ausgeflossen, welche man mit dem Namen *prae-Etna-Laven* belegt hat. (Nº. 1, 2, 3, Tabelle I). Das Urteil der kompetenten Kenner des Etnas über den Zusammenhang dieser Gesteine mit denjenigen des eigentlichen Etnas ist verschieden. Aus der von mir in diesem Artikel gegebenen Interpretation dieser Gesteine ist ersichtlich, dass diese drei Gesteine zwar Uebereinstimmung zeigen mit den eigentlichen Gesteinen des Etna, dass jedoch die Nummern 2 und 3, namentlich 2, sehr deutliche Abweichungen zeigen. Nº. 1 ist dagegen ein typisches Etnagestein.

Die eigentlichen Eruptionen des Etna fingen an in der quaternären Catanischen Bucht. Diese ersten Laven haben den Namen Sockel- oder

<sup>1)</sup> W. F. GISOLF. Zur graphischen Darstellung von Gesteinsanalysen. T.M.P.M., 1928, 39, S. 76.

Terrassenlaven empfangen. Danach folgten die Eruptionen, welche den Vulkankegel aufbauten bis zum heutigen Tage<sup>1)</sup>.

In Tabelle I sind diese Gesteine mit ihren Nigglichen, von mir etwas modifizirten, Werten aufgenommen, zusammen mit den Parametern eines Amphibols und eines Pyroxens, welche aus Etnalaven stammen und wovon gute Analysen bekannt sind. Die Analysen dieser zwei Mineralien und der Gesteine №. 1—№. 23 sind dem zitierten Artikel Washingtons c.s. entnommen; die übrigen Nummern sind den bekannten Analysentabellen Washingtons (1917) entlehnt.

Vor der Berechnung der Nigglichen Zahlen habe ich in den Molekularquotienten die Accessorien, ausgenommen Magnetit und Ilmenit, in Abzug gebracht, damit man ein genaueres Bild der Silikatgemische erhielte. Niggli unterscheidet fünf Parameter: *si*, *fm*, *al*, *c*, *alk* und zwei Quotienten *k* und *mg*; an deren Stelle nehme ich sieben Parameter: *si*,  $2\text{fe}_2\text{o}_3$  *feo*, *mgo*, *al*, *c*, *na*, *ka* in der Weise, dass  $\text{fm} = 2\text{fe}_2\text{o}_3 + \text{feo} + \text{mgo}$ ;  $\text{alk} = \text{na} + \text{ka}$ . Diese Symbole sind deutlich genug, um keiner Erklärung zu bedürfen.

Wenn man die Nigglichen Parameter *fm*, *c*, *al*, *alk* und *si* der genannten Gesteine und Mineralen einträgt in der von mir in dem auf Seite 1 zitierten Artikel<sup>2)</sup> gegebenen *alkal*-Projektion des Tetraeders mit Kieselsäurefeld, so fällt sofort auf, dass im *alkal*-Felde die Projektionen sämtlich konvergieren nach der Projektion  $A_1$  des Amphibols. Der weitaus grösste Teil der Projektionen fällt innerhalb  $\triangle A_1 P_1 Q_1$  (Fig. 1). Ausgenommen sind die Gesteine vom Val di Noto und die Nummern 2 and 3 der *prae*-Etna-Laven, während von den Etnalaven die Nummern 25 und 20 ausserhalb des genannten Dreiecks liegen<sup>3)</sup>.

Die Punkte der Geraden  $P_1 Q_1$  sind Projektionen von Feldspatgemischen, falls ihre Kieselsäurepole auf der Geraden  $P_3 Q_3$  gelegen sind. Ist dies der Fall, so kann irgend ein Gestein aufgefasst werden als heteromorphes Gestein eines Amphibolplagioklasgemisches mit Amphibol  $A_1$ , d.h. als ein heteromorphes Gestein eines Diorits oder Gabbros. Tatsächlich sind die Etnalaven aufgebaut aus Plagioklas, Augit,  $\pm$  Olivin, Magnetit, in einzelnen Fällen von Amphibol oder Nephelin begleitet.

Diese Auffassung, andesitisch-basaltische Gesteine zu betrachten als Äquivalente eines gabbronoritischen Magmas mit Amphibol-Plagioklas-Zusammensetzung hat ein grosser Reiz; einerseits, weil man voraussetzen darf, dass unter gewissen Umständen diese Gesteine als Amphibol-plagioklasgesteine hätten kristallisiren können; anderseits, weil die Metamorphose solche Gesteine in Amphibolite umzuwandeln pflegt.

Man kann für jedes Gestein bestimmen: 1<sup>o</sup>. die Zusammensetzung des Amphibols und 2<sup>o</sup>. die Zusammensetzung des Plagioklases in dem hetero-

<sup>1)</sup> Diese Daten des Etna sind dem Artikel von WASHINGTON, AUROUSSEAU und KEYES "Die Laven des Etna". Am. J. of Sc. 1926, 12, 371 entnommen.

<sup>2)</sup> S.a. diese Sitzungsberichte, Sitzung von 29 Okt. 1927, p. 1030 (holl. Ausgabe).

<sup>3)</sup> №. 27 ist unterdrückt, da es sich um verwehter Basaltsand handelt. №. 24 fällt zusammen mit №. 6; №. 30 mit №. 29.

morphismen Diorit oder Gabbro in folgender, für N°. 8 beschriebenen, Weise:

Ziehe im *alkal*-Felde (Fig. 1) die Gerade  $A_1 8$ , bis sie die Feldspatlinie  $P_1 Q_1$  in  $P_1$  schneidet. Ziehe durch  $P_1$  einen Ordinaten  $P_2 P_1 P_3$ , welche die Feldspatlinien im *cfm*-Felde und im *si*-Felde in  $P_2$  bzw.  $P_3$  schneidet. Ziehe im *si*-Felde durch  $P_3$  und 8, eine Gerade  $P_3 8$ , welche die Amphibollinie  $A_2 A_1$  in  $A'_3$  schneidet.  $A'_3$  ist dann der *si*-Pol des Amphibols des mit N°. 8 heteromorphen Diorits. Desgleichen tut man im *cfm*-Felde und findet dann den *cfm*-Pol des Amphibols ein wenig oberhalb  $A'_2$ . In Fig. 2 kann man in ähnlicher Weise den Parameter *mgo* bestimmen. (Der Uebersichtlichkeit wegen sind die Figuren 1 und 2 getrennt gehalten; nichts steht jedoch einer Vereinigung dieser zwei Figuren im Wege.) Der Parameter *mgo* des Amphibols in N°. 8 ist in Fig. 2 die von  $a''_2$  auf die obere Seite des Quadrats gefällte Lotlinie<sup>1)</sup>.

Der in Tabelle I erwähnte Amphibol A wird angegeben durch die drei Projektionen  $A_1, A_2, A_3$ . Man kann diesen Amphibol normativ auffassen als Magnetit + Olivin + Diopsid + Anorthit + Nephelin. Falls dieser Nephelin silifiziert wird zu Albit, dann verschiebt sich der Kieselsäurepol des Amphibols nach  $A'_3$ ; während er in  $A'_3$  liegt, wenn die übrigen Bestandteile zu Metasilikat silifiziert werden.

Die Zusammensetzung des Plagioklases im Diorit wird, für das Gestein N°. 8, durch den Punkt  $P_1$  bestimmt. Im heteromorphen Diorit ist also das Verhältniss Alkalifeldspat: Anorthit =  $An P_1 : P_1 Alk$ . (Man muss die Beschränkung der Schwerpunktsregel dabei im Auge behalten, nml., dass alle Mineralmoleküle mit der gleichen Anzahl basischer Oxyde geschrieben werden müssen). In Fig. 2 kann auch der Kaliumgehalt des Alkalifeldspats bestimmt werden, insoferne als man den Amphibolpol  $a_1$  mit 8 verbindet bis zum Schnitt  $p_1$  mit  $P_1 P_2$ ; das Verhältniss der Strecken, worin  $p_1$  die Senkrechte von  $P_1$  auf die untere Seite des Quadrats teilt, ergibt das molekulare Verhältniss des Albites zum Orthoklas im heteromorphen Diorit. Ausserdem ist im *alkal*-Felde das Verhältniss der Strecken  $P_1 8$  und  $8 A_1$  dem molekularen Verhältnisse des Amphibols zum Plagioklase gleich<sup>2)</sup>. Die obengenannte Beschränkung des Schwerpunktssatzes ist dabei wiederum zu berücksichtigen.

Wenn man nun für alle Gesteine die Zusammensetzung des Amphibols und des Plagioklases im heteromorphen Diorit, entweder graphisch, oder mittels Berechnung bestimmt, dann kommt man zu den Resultaten der

<sup>1)</sup> Man kann die Parameter  $2 fe_{2O_3}$ , *feo*, *mgo*, *c* des Amphibols auch berechnen mit den einfachen Formeln:

$$(mgo)_1 = \frac{50 - al_1}{50 - al} \times mgo; \quad (2 fe_{2O_3})_1 = \frac{50 - al_1}{50 - al} \times 2 fe_{2O_3}; \quad (feo)_1 = \frac{50 - al_1}{50 - al} \times feo;$$

$$c_1 = 100 - (al_1 + alk_1 + fm_1).$$

Die mit dem Index 1 versehenen Parameter sind diejenigen des Amphibols; die übrigen Parameter gehören dem Gestein an.

<sup>2)</sup> S.a. meine Mitteilung in diesen Sitzungsberichten, Sitzung vom 28. Januar 1928, p. 57 (holl. Ausgabe).

W. F. GISOLF: DIE LAVEN VOM ETNA.

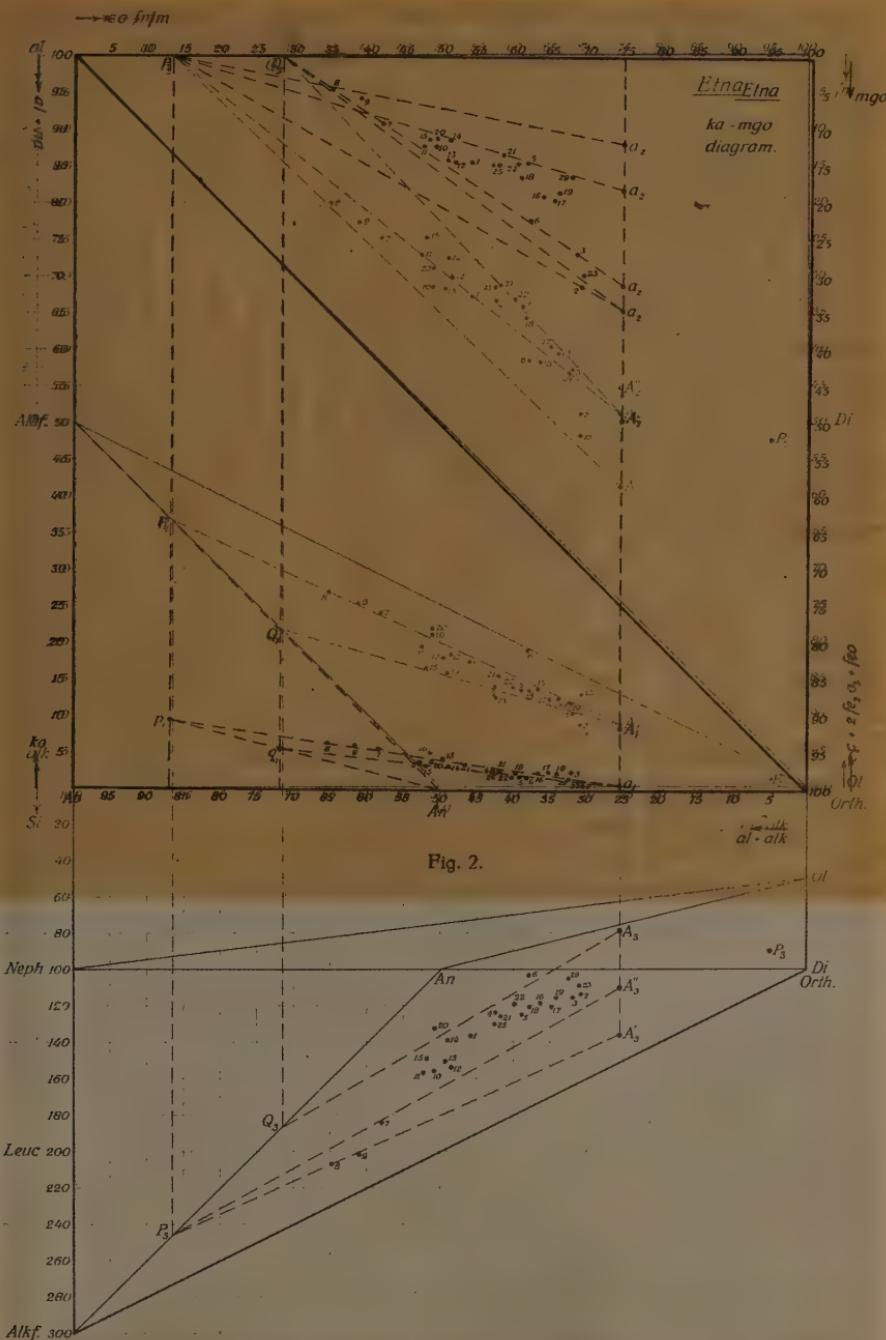


Fig. 1.

morphismen Diorit oder Gabbro in folgender, für N°. 8 beschriebenen Weise:

Die Ziege im *alkal.-Feld*: (Fig. 1) die Gerade  $A_1$  8 bis sie die Feldspatlinie  $P_1Q$  in  $P_1$  schneidet. Ziege durch  $P_1$  einen Quadranten  $P_2P_3P_4$ , welche die Feldspatlinien im *cfc*-Feld und im *st*-Feld in  $P_2$  bzw.  $P_3$  schneidet. Ziege im *st*-Feld durch  $P_3$  und  $8$  Gerade  $A_2$  8, welche die Amphibollinie  $A_2A_3$  in  $A_3$  schneidet.  $A_2$  ist dann der *st*-Pol des Amphibols des mit N°. 8 heteromorphen Diorits. Desgleichen tut man im *cfc*-Feld und findet dann den *cfc*-Pol des Amphibols ein wenig oberhalb  $A'_2$ . In Fig. 2 kann man in ähnlicher Weise den Parameter *mgo* bestimmen. (Der Übersichtlichkeit wegen sind die Figuren 1 und 2 getrennt gehalten; nichts steht jedoch einer Vereinigung dieser zwei Figuren im Wege.) Der Parameter *mgo* des Amphibols in N°. 8 ist in Fig. 2 die von  $a''_2$  auf die obere Seite des Quadrats gefällte Lotlinie<sup>1)</sup>.

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Die Zusammensetzung des Plagioklases im Diorit wird, für das Gestein N°. 8, durch den Punkt  $P_1$  bestimmt. Im heteromorphen Diorit ist also das Verhältniss Alkalifeldspat: Anorthit =  $An P_1 : Alk f.$  (Man muss die Beschränkung der Schwerpunktsregel dabei im Auge behalten, nml., dass alle Mineralmoleküle mit der gleichen Anzahl basischer Oxyde geschrieben werden müssen). In Fig. 2 kann auch der Kaliumgehalt des Alkalifeldspats bestimmt werden, insoferne als man den Amphibolpol  $a_1$  mit 8 verbindet bis zum Schnittpunkt  $P_2$  des Verhältniss der Strecken  $P_1 8$  und  $8 A_1$  die Senkrechte von  $P_1$  auf die untere Seite des Quadrats teilt, ergibt das molekulare Verhältniss des Albits zum Orthoklas im heteromorphen Diorit. Ausserdem ist im *alkal.-Feld* das Verhältniss der Strecken  $P_1 8$  und  $8 A_1$  dem molekularen Verhältnisse des Amphibols zum Plagioklase gleich<sup>2)</sup>. Die obengenannte Beschränkung des Schwerpunktssatzes ist dabei wiederum zu berücksichtigen.

Wenn man nun für alle Gesteine die Zusammensetzung des Amphibols und des Plagioklases im heteromorphen Diorit, entweder graphisch, oder mittels Berechnung bestimmt, dann kommt man zu den Resultaten der

1) Man kann die Parameter  $2fe_{203}$ ,  $feo$ ,  $mgo$ ,  $c$  des Amphibols auch berechnen mit den einfachen Formeln:

$$(mgo)_1 = \frac{50 - al_1}{50 - al} \times mgo; \quad (2fe_{203})_1 = \frac{50 - al_1}{50 - al} \times 2fe_{203}; \quad (feo)_1 = \frac{50 - al_1}{50 - al} \times feo;$$

$$c_1 = 100 - (al_1 + alk_1 + fm_1).$$

Die mit dem Index 1 versehenen Parameter sind diejenigen des Amphibols; die übrigen Parameter gehören dem Gestein an.

2) S.a. meine Mitteilung in diesen Sitzungsberichten, Sitzung vom 28. Januar 1928, p. 57 (holl. Ausgabe).

W. F. GISOLF: DIE LAVEN VOM ETNA.

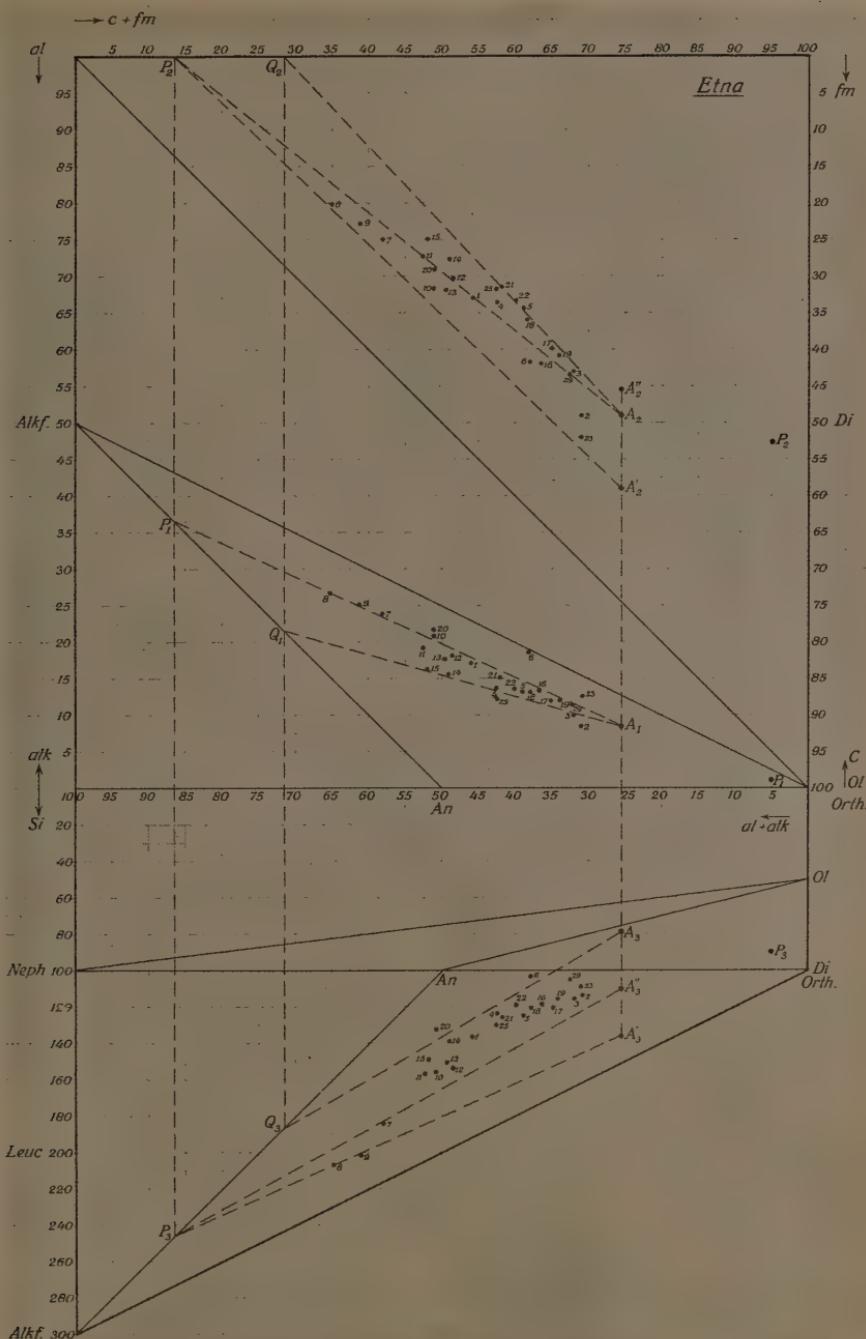


Fig. 1.



TABELLE 1.

N. <sup>o</sup>	Namen	Alter	si	2 Fe <sub>2</sub> O <sub>3</sub>	FeO	MgO	C	Al	Na	Ka	Rundort
P	Pyroxen	1669	90	5.8	9.5	37.2	42.5	3.8	1.2	—	Monte Rossi
A	Amphibol	praehistoric	78	7.0	7.9	34.1	25.5	16.9	7.7	0.9	Milo; Ostabhang
1	Andesinandesit	prae-Etnaisch	137	5.5	13.6	14.1	20.9	28.9	13.3	3.7	Grotta della Palombe
2	Andesinbasalt	"	114	7.1	10.9	31.1	20.0	22.5	7.5	0.9	Aci Castello
3	Olivinlabradorbasalt	"	116	4.3	12.3	26.7	24.7	22.0	7.6	2.4	Sorgente Jusu
4	Andesinandesit	Terrassen lava	124	4.8	14.2	14.5	23.9	29.1	10.6	2.9	Station Paterno
5	Andesinandesit	"	123	6.3	14.1	14.1	26.7	25.8	10.8	2.2	Valle delle Petrole
6	Basalt	miozän	104	6.6	13.3	21.9	20.4	19.6	16.2	2.0	Val di Noto
7	Oligoklasandesit	praehistoric	185	4.6	11.5	9.0	16.6	34.6	18.0	5.7	Val dell' Acqua
8	"	"	207	4.7	11.3	4.3	14.7	38.4	20.2	6.4	Val del Monaco
9	"	"	202	8.8	8.5	5.4	16.0	36.4	18.7	6.2	"
10	"	"	156	5.9	13.8	12.0	17.2	30.4	15.5	5.2	Val del Bove
11	Oligokl. andasianandesit	"	157	7.9	7.6	11.8	20.1	33.4	15.1	4.1	"
12	Andesinandesit	"	154	5.4	11.0	13.8	21.2	30.6	14.4	3.6	"
13	"	"	151	11.9	6.0	14.1	18.6	31.9	13.1	4.4	"
14	"	"	139	3.3	13.3	11.1	23.2	33.7	11.8	3.6	"
15	"	"	149	2.5	11.4	11.0	23.1	36.1	12.4	3.5	"
16	Andesinbasaltandesit	"	119	12.9	10.1	18.7	21.6	23.3	10.8	2.6	Punta Lucia
17	Basalt	1669	121	5.3	15.8	19.0	25.0	23.0	9.5	2.4	Catania
18	Andesinandesit	1792	121	5.4	14.6	15.8	24.8	26.1	10.7	2.6	Val del Bove
19	Andesinbasalt	1910	117	6.1	16.6	18.3	24.4	22.7	9.4	2.5	Oberhalb Nicolosi
20	Nephelinoligoklasandesit	1329	133	5.4	12.4	11.1	20.3	29.3	17.8	3.7	Monte Rossi
21	Andesinbasalt	1908	125	5.4	13.2	13.0	26.3	27.0	12.1	3.0	?
22	"	1910	120	5.3	13.9	14.0	26.6	26.5	11.5	2.2	Monte Recupero
23	Basaltglas	miozän	109	9.6	12.7	29.5	17.4	18.1	11.4	1.3	Val di Noto
24	"	"	102	1.3	19.9	20.6	21.1	19.2	16.1	1.8	"
25	Basalttasche	1911	130	10.5	6.5	14.4	26.1	29.9	9.4	3.2	Catania
27	Basaltstand	1911	122	10.9	12.7	14.1	18.9	25.7	14.2	3.5	
28	Basalt	1908	121	10.1	9.4	12.8	25.7	27.2	11.8	3.0	
29	"	"	105	17.7	16.4	24.4	20.8	9.6	2.0		

Tabelle 2. Es ist jedoch unmöglich die Gesteine des Val di Noto und des N°. 20 als heteromorphe Gesteine eines Diorits zu betrachten, falls ein Amphibol mit denselben *al*- und *alk*-Zahlen daran teilnehmen sollte. Die Zahl der Analysen dieser Gesteine ist jedoch zu klein um sie zu diskutieren.

Bevor übergegangen wird zu der Besprechung dieser Resultate der Tabelle 2, sei zuerst aufmerksam gemacht auf die merkwürdige Tatsache, dass die Amphibole in fast allen heteromorphen Dioriten (oder Gabbros) dieselben Parameter *al* und *alk*, und, wie aus Fig. 2 ersichtlich ist, auch fast alle dieselben Parameter *na* und *ka*, haben. Diese Tatsache suggeriert den Gedanken, dass ein bestimmtes Ion im Magma vorhanden gewesen sein muss. Tatsächlich hat dieses Ion die Formel  $A_1 T_2 M_9$  ( $A = Na_2O + K_2O$ ,  $T = Al_2O_3$ ,  $M = CaO + MgO + 2Fe_2O_3 + FeO$ ). Diesem Ion entsprechen die Niggliischen Zahlen  $c + fm = 75$ ,  $al = 16.7$   $alk = 8.3$  statt der aus der Analyse berechneten Werten  $c + fm = 74.5$ ,  $al = 16.9$ ,  $alk = 8.6$ . Dem gegebenen Amphibol würde dann die Formel  $A_1 T_2 M_9 (SiO_2)_9$  entsprechen (mit  $si = 75$  statt 78), während den Amphibolen mit den Kieselsäurepolen  $A''_3$  und  $A'_3$  die Formeln  $A_1 T_2 M_9 (SiO_2)_{13}$  bzw.  $A_1 T_2 M_9 (SiO_2)_{16}$  entsprechen würden. Ob dieser letzte Amphibol noch bestehen könnte mit seinem hohen Kieselsäuregehalt ist eine offene Frage.

Zur Besprechung der Resultate in Tabelle 2 übergehend, sei zuerst bemerkt, dass für die Amphibole die Summe  $c + mgo + 2fe_2O_3 + feO$  konstant ist. Falls wir die zwei Eisenoxyde zusammenfügen, können wir die Zusammensetzung dieser Amphibole graphisch darstellen mittels eines gleichseitigen Dreiecks, was in Fig. 3 geschehen ist. Man sieht, dass alle Amphibolpole dicht bei einander liegen mit Ausnahme der Pole der Gesteine 8 und 9 einerseits, und der Gesteine 2 und 3 anderseits. Die Gesteine 2 und 3 gehören zu den ältesten Gesteinen und die Gesteine 8 und 9 zu den nächstjüngeren Laven des Etna. Die Gesteine 2 und 3 gehören zu den basischen Differentiationsprodukten, die Gesteine 8 und 9 zu den sauren Spaltungsprodukten des Magmas. Dabei ist ersichtlich, dass die Amphibole der sauren Spaltungen sehr eisenreich, die Amphibole der basischen Spaltungen dagegen sehr magnesiumreich sind; eine Erscheinung, die in mehreren Vulkangruppen auftritt, wie ich später ausführen werde.

Ausgenommen die Gesteine 7, 8, 9 und 2 würden alle Diorite und Gabbros, welche den Etnalaven entsprechen, einen Amphibol besitzen, deren Parameter *si* gleich oder kleiner ist als 108, d.h. alle Gesteine, diejenigen mit  $si \geq 108$  nicht mitgerechnet, werden normativen Nephelin besitzen. Derartige Gesteine könnten durch den Zerfall des positiven Amphibolions in nephelinhaltige Gesteine übergehen, wofür dem Olivin nicht Kieselsäure unter Erzabscheidung entzogen würde, damit der Nephelin wiederum zu Albit silifiziert würde. In diesem Zusammenhang darf wohl die Aufmerksamkeit auf die vielfältig vorkommenden Uebergänge von Amphibol in Opazit in Effusivgesteinen gelenkt werden.

TABELLE 2.

Nr.	Parameter der Amphibole				% An der Plag. <sup>1)</sup>	Verhältniss Plag. Amph.
	mgO	2[Fe <sub>2</sub> O <sub>3</sub> ]	c	si		
1	22.1	30.0	22.4	85	36	36 : 64
2	37.4	21.5	15.6	110	85	17 : 83
3	31.6	19.6	23.3	106	65	15 : 85
4	23.0	30.1	21.4	87	56	37 : 63
5	19.3	27.9	27.2	95	50	27 : 73
7	19.3	34.6	20.6	110	27	53 : 47
8	12.3	45.7	16.5	136	27	65 : 35
9	13.1	42.1	19.3	140	27	59 : 41
10	20.3	33.3	20.9	90	23	41 : 59
11	23.5	30.9	20.1	92	40	50 : 50
12	23.5	28.0	23.0	102	37	41 : 59
13	25.8	32.7	16.0	98	44	45 : 55
14	22.5	33.7	18.3	89	56	51 : 49
15	26.2	33.1	15.2	95	58	58 : 42
16	23.2	28.5	22.8	92	33	19 : 81
17	23.3	25.9	25.3	100	47	18 : 82
18	21.9	27.7	24.9	92	43	28 : 72
19	22.2	27.5	24.8	95	45	18 : 82
20	(27.5)	(24.0)	(23.0)	(50)	(14)	(37 : 63)
21	18.7	26.8	29.0	83	40	31 : 69
22	19.7	27.0	27.8	84	48	29 : 71
25	23.7	28.0	22.8	102	62	39 : 61
28	18.6	28.3	27.6	80	43	31 : 69
29	18.6	30.4	25.5	87	32	12 : 88
30	19.7	30.7	24.1	87	31	12 : 87
A	34.1	14.9	25.5	78	—	—

Der Anorthitgehalt der Plagioklase und das Verhältniss Plagioklas : Amphibol lassen sich in ihrem Zusammenhange am besten graphisch über-

<sup>1)</sup> Die Plagioklase sind gedacht als  $m \text{ CaO} \cdot \text{Al}_2\text{O}_3 \cdot 2\text{SiO}_2 + n(\text{Na}_2\text{O} \cdot \text{K}_2\text{O}) \cdot \text{Al}_2\text{O}_3 \cdot 6\text{SiO}_2$ .

blicken. Das ist in Fig. 4 geschehen. Man ersicht aus dieser Figur, dass mit den Gesteinen 8 und 9 am meisten die Gesteine 7 und 10 überein-

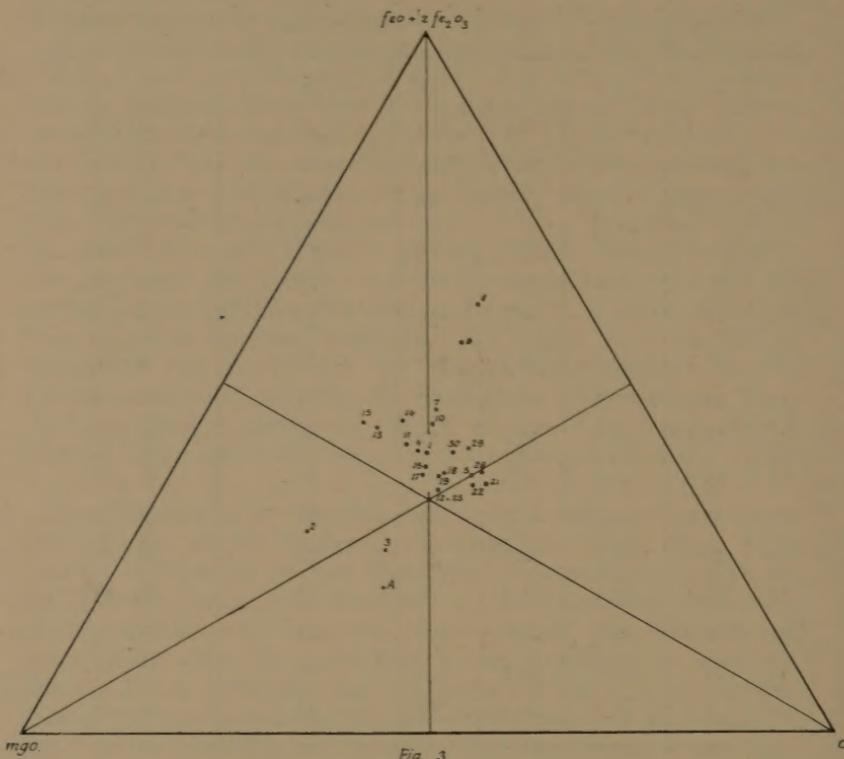


Fig. 3.

stimmen; und dass, wie in Fig. 3, auch die Gesteine 2 und 3 isolirt auftreten. Die Gesteine 17 und 19 zeigen mit ihnen noch die meiste Ähnlichkeit. Weiter sind die Gesteine 29, 30, 16 einander ziemlich ähnlich; die übrigen Gesteinspole liegen zerstreut und lassen vermuten, dass es unter den Etnalaven noch mehrere Gesteinstypen gibt.

Beim Lesen der Figur 4 sei man der mehrgenannten Beschränkung des Schwerpunktssatzes eingedenkt, nml. 1<sup>0</sup>. dass die verschiedenen Feldspatmoleküle eine gleiche Anzahl basischer Oxyde enthalten müssen (35 % An bezieht sich also auf einen Plagioklas mit der mineralogischen Formel  $Ab_{130} An_{35}$  oder  $Ab_{79} An_{21}$ . Orthoklas zu Albit gerechnet); 2<sup>0</sup>. dass die Anweisung Plag. %, sich bezieht auf Mineralgemische, welche derselben Beschränkung gehorchen. Z. B. Gestein N<sup>o</sup>. 16 hat einen Plagioklas mit 33 % An, d.h. mit der mineralogischen Formel  $Ab_{80} An_{20}$ . Das Verhältniss Plagioklas : Amphibol = 19 : 81; also sind vorhanden:  $19 \times \{0.06 \times 67\}$  Moleküle  $[(Na_2O, K_2O) \cdot Al_2O_3 \cdot 6 SiO_2]$

$+ 0.06 \times 33$  Moleküle  $[\text{CaO} \cdot \text{Al}_2\text{O}_3 \cdot 2\text{SiO}_2]$ } gegen 81 Moleküle  $\text{A}_1 \text{T}_2 \text{M}_9 (\text{SiO}_2)_{11}$ .

Die Zahl der basischen Oxyde im Plagioklas ist  $0.12 \times (67 + 33) = 12$  und im Amphibol  $1 + 2 + 9 = 12$ , womit der Beschränkung des Schwerpunktssatzes genüge getan ist.

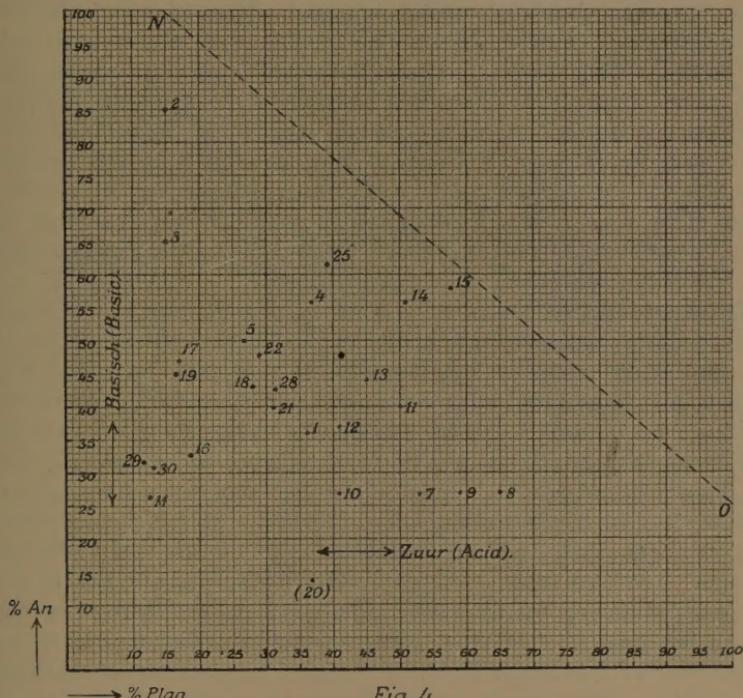


Fig. 4.

Fig 4 ist besonders geeignet zum Studium der Spaltungen im Etnamagma. Alle Gesteinspole liegen im Dreieck MNO, während die ältesten Gesteine 2, 3, 7, 8, 9 und 10 auf den Katheten dieses Dreiecks gelegen sind und zugleich die basischen und sauren Spaltungsprodukte andeuten. Ein Gestein mit der Zusammensetzung M würde sowohl zur basischen wie zur sauren Gruppe gehören. Die übrigen Gesteine können betrachtet werden als Gemische dieser äussersten Gesteine oder umgekehrt: die äussersten Gesteine, deren Pole auf den Katheten MN und MO gelegen sind, können als Spaltungsprodukte des Magmas der übrigen Gesteine aufgefasst werden.

Merkwürdig ist die Tatsache, dass die Amphibole der Gesteine 2, 3, 7, 8, 9 übersättigt sind mit Kieselsäure, während die Amphibole der übrigen Gesteine untersättigt oder gerade gesättigt sind. Man würde, mit DALY, diese Tatsache einem Kontakt etwa mit Kalkstein in späterer Zeit zuschreiben können; vielleicht könnte das auch eine Erklärung sein für die

Genese des N°. 20, das von den übrigen Gesteinen bedeutend abweicht<sup>1)</sup>.

Es ist deutlich, dass bei dem Gesteine N°. 20, das auch in Tabelle 2 und in Fig. 4 aufgenommen ist, jedoch zwischen Klammern um den abweichenden Charakter zu betonen, die *si*-Zahl des Amphibols zu niedrig ist, als dass er noch als solcher bestehen könnte. In diesem Gesteine muss also magmatischer Nephelin entstanden sein, auch wenn man voraussetzen sollte, dass nicht Amphibol, sondern Olivin, der Pyroxen P und Magnetit die melanokraten Gemengteile des Tiefengesteins sein würden. Man kann das in Fig. 1 leicht beweisen. Diese Eigenschaft magmatischen Nephelin (d.h. keinen Nephelin, welche man entstanden denken könnte aus Spaltung von positiven Amphibolionen) zu besitzen, teilt N°. 20 mit den Gesteinen des Val di Noto, 23, 24, 6. Daher vermute ich, dass diese vier Gesteine von Magmas herkommen, welche unter gleichen Bedingungen gestanden haben. Bevor wir jedoch zu Diskussionen übergehen können, müssen wir die von WASHINGTON in Aussicht gestellten weiteren Analysen von Gesteinen des Val di Noto abwarten.

Zusammenfassend können wir also sagen, dass die eigentlichen Eruptionen des Etna anfingen mit den basischen Spaltungsprodukten (2 und 3); dass danach die sauren Spaltungsprodukte zu Tage gefördert worden sind (7, 8, 9, 10); dass diese beiden Gesteinsreihen mit Kieselsäuren übersättigt worden sind; und dass nachher das eigentliche Magma, das inzwischen kieselsäure-ärmer geworden war, in vorhistorischer und historischer Zeit ans Tageslicht gebracht worden ist. Unter den historischen Laven gibt es auch wieder basische Spaltungsprodukte (17, 19, 29, 30) welche sich jedoch von den *prae*-Etna-Laven unterscheiden: 1°. durch eine geringere Sättigung mit Kieselsäure; 2°. durch särkere Plagioklase in den heteromorphen Dioriten. Auch unter den vorhistorischen Laven gibt es ein einzelnes derartiges Gestein (16). Eine merkwürdige Ausnahme bildet die Lave von 1329, von Monte Rossi bei Fleri, welche grosse Ähnlichkeit mit den miozänen Laven von Val di Noto aufweist.

Die mittlere Zusammensetzung des Etnamagmas zu besprechen hat keinen Sinn; einerseits weil es wahrscheinlich noch mehrere nicht analysirte Gesteinstypen gibt; anderseits weil man dazu die sichtbaren und unsichtbaren Gesteinsmengen jedes Typus kennen müsste, was jedoch unmöglich ist.

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1) S.a. die Mitteilung von H. A. BROUWER über derartige Prozesse in Gesteinen vom Vulkan Merapi auf Java. Diese Sitzungsberichte, Sitzung vom 25. Februar 1928.